ON KIM-INDEPENDENCE

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Abstract. We study $NSOP_1$ theories. We define *Kim-independence*, which generalizes non-forking independence in simple theories and corresponds to non-forking at a generic scale. We show that Kim-independence satisfies a version of Kim's lemma, symmetry, and an independence theorem and that, moreover, these properties individually characterize $NSOP_1$ theories. We describe Kim-independence in several concrete theories and observe that it corresponds to previously studied notions of independence in Frobenius fields and vector spaces with a generic bilinear form.

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1. Introduction

The class of simple theories was one of the first classes of unstable theories to receive extensive study. The starting point is *Classification Theory* [She90], where, in the course of studying stable theories, Shelah isolates *local character* as a key property of non-forking independence and observes a dichotomy in the way local character can fail, a theorem we now recognize as saying that a non-simple theory must have the tree property of the first or second kind. Shortly after the publication of the first edition of [She90], Shelah defined the class of simple theories and characterized them in terms of a certain chain condition of the Boolean algebra of non-weakly dividing formulas, which in turn led to consistency results on their saturation spectra [She80]. The aim of that work was to obtain an 'outside' set-theoretic definition of the class to support the claim that simplicity marked a dividing line. In separate developments, questions concerning concrete examples created the need for new methods to treat unstable structures. Hrushovski and Pillay used local stability and S_1-rank in the study of the defininability of groups in pseudo-finite and PAC fields in [HP94], and these methods were situated in the broader context of

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PAC structures studied by Hrushovski [Hru91], where an independence theorem was proved. Moreover, Lachlan's far-reaching theory of smoothly approximated structures furnished examples of tame unstable theories. After Kantor, Liebeck, and Macpherson [KLM89] classified the primitive smoothly approximable structures, Cherlin and Hrushovski [CH03] used stability theoretic methods concerning independence and amalgamation to describe how these primitive pieces fit together to form a quasi-finite structure.

Kim's thesis and subsequent work by Kim and Pillay showed how to regard these developments as instances of a common theory, with non-forking independence at its center. Kim and Pillay proved that in a simple theory, forking and dividing coincide, non-forking independence is symmetric and transitive, and that the independence theorem holds over models. Moreover, Kim showed that symmetry and transitivity of non-forking both individually *characterize* the simple theories, and Kim and Pillay showed that any independence relation satisfying the basic properties of non-forking independence must actually coincide with non-forking independence, giving both a striking characterization of the simple theories and a powerful method for showing that a particular theory is simple, namely by observing that it has an independence relation of the right kind.

Here, we study the class of NSOP_1 theories. These are the theories which do not have the property SOP_1, which form a class of theories that properly contain the simple theories and which are contained inside the class of theories without the tree property of the first kind. SOP_1 was defined by Shelah and Dzamonja in their study of the $\trianglelefteq^*$-order [DS04] and later studied by Shelah and Usvyatsov in [SU08]. The NSOP_1 theories were characterized as the theories satisfying a weak independence theorem for invariant types by Chernikov and the second-named author in [CR16]. This characterization provided a point of contact between the combinatorics of model-theoretic tree properties and the study of definability in particular algebraic examples. Chatzidakis [Cha99], [Cha02] studied independence in ω-free PAC fields and, more generally, Frobenius fields and showed that the independence theorem holds for these structures even though they are not simple. Similarly, Granger showed in his thesis that the model companion of the theory of infinite-dimensional vector spaces with a bilinear form is not simple but nonetheless comes equipped with a good notion of independence. The amalgamation criterion of [CR16] established that these structures have NSOP_1 theory by appealing to the existence of these independence relations, but what was missing was a theory of independence in NSOP_1 theories more generally. The purpose of this paper is to establish exactly such a theory.

One central tool in the study of forking in simple theories is Kim's lemma: a formula divides over a set A if and only if it divides with respect to some Morley sequence over A if and only if it divides for all Morley sequences over A. In [CK12], this was shown to hold over models in NTP_2 theories, provided that the Morley sequence is a strict invariant Morley sequence. In the setting of NSOP_1 theories, we find a new phenomenon: forking which is *never* witnessed by a generic sequence. In fact, we show that any NSOP_1 theory with a universal witness to dividing must be simple (Proposition 7.6 below) and that forking need not equal dividing in an NSOP_1 theory. Nonetheless, we find that, by restricting attention to the forking that is witnessed by a generic sequence, one can recover many of the properties of forking in simple theories. We show moreover that this kind of simplicity at a generic scale is characteristic of NSOP_1 theories.

There is considerable freedom in the choice of notion of generic sequence. One suggestion which inspired our work is due to Kim, who proposed in his 2009 talk on NTP_1 theories [Kim09] that one might develop an independence theory for NTP_1 theories or a subclass therein by considering only formulas which divide with respect to every non-forking Morley sequence. Compared to invariance or finite satisfiability, forking is a relatively weak notion of independence and this notion proved unwieldy at the beginning stages of developing the theory presented here. However, Hrushovski's study of q-dividing [Hru12] and Malliaris and Shelah's characterization of NTP_1 theories in terms of higher formulas [MS15] provided evidence that one might be able to build a theory around an investigation of formulas that divide with respect to a Morley sequence in a global invariant or finitely satisfiable type. Building off this work, we introduce the notion of Kim-dividing – a formula *Kim-divides* over a set A if it divides with respect to a Morley sequence in a global A-invariant type – and the associated notion of independence, *Kim-independence*. Our first observation is that a theory is NSOP_1 if and only if Kim-dividing satisfies a version of Kim's lemma over models, where a formula divides with respect to a Morley sequence in *some* global invariant type extending the type of the parameters if and only if it divides with respect to *every* Morley sequence in an appropriate invariant type.

From Kim's lemma for Kim-dividing, many familiar properties of non-forking independence follow: Kim-forking equals Kim-dividing, Kim-independence satisfies extension and a version of the chain condition, etc. In Section 4, we show additionally that Kim-independence is symmetric over models. The argument there centers upon the notion of a *tree Morley sequence* which is defined in terms of indiscernible trees. We show that tree Morley sequences always witness Kim-dividing and prove a version of the chain condition for them. In Section 5, we prove the independence theorem. In Section 7, we state our main theorem: Kim's lemma for Kim-dividing, symmetry over models, and the independence theorem both hold in NSOP_1 theories and individually characterize NSOP_1 theories. We also show that the simple theories can be characterized in several new ways in terms of Kim-independence. In particular, we show that Kim-independence coincides with non-forking over models if and only if the theory is simple, which means that our theorems imply the corresponding facts for non-forking independence in a simple theory. Finally, we prove that in an NSOP_1 theory a formula Kim-divides over a model if and only if it divides with respect to every non-forking Morley sequence in the parameters and this too characterizes NSOP_1 theories. This means that Kim-independence could have been defined from the outset in essentially the way Kim proposed, but curiously, proving anything about this notion without making use of invariant types seems quite difficult.

We conclude the paper with Section 8 where we describe Kim-independence explicitly in several concrete examples. We show it may be described in purely algebraic terms in the case of Frobenius fields, where Kim-independence turns out to coincide with *weak independence*, as defined by Chatzidakis. We also show that in Granger's two-sorted theory of a vector space over an algebraically closed field with a generic bilinear form, Kim-independence is the same as Granger's Γ-independence for singletons and may be given a clean algebraic description in general. These results suggest the naturality and robustness of Kim-dividing, but also serve to explain the simplicity-like phenomena observed in these concrete examples on the basis of a general theory. In this section, we additionally describe a combinatorial example of a NSOP$_1$ theory, based on a variant of T^*_{feq} introduced by Džamonja and Shelah, which furnishes counter-examples to some *a priori* possible strengthenings of the results we prove. In particular, we give the first example of a simple non-cosimple type, answering a question of Chernikov [Che14].

2. Syntax

In this section we will define SOP$_1$ and prove its equivalence with a syntactic property of a different form. This will allow us to relate SOP$_1$ to dividing. We will often work with arrays and trees. Suppose $(c_{ij})_{i<\kappa, j<\lambda}$ is an array. Write $\overline{c}_i = (c_{i,j})_{j<\lambda}$ for the ith row of the array and $\overline{c}_{<i}$ for the sequence of rows with index less than i, i.e. $(\overline{c}_k)_{k<i}$. Suppose $\mathcal{T}$ is a tree, $(a_\eta)_{\eta\in\mathcal{T}}$ is a collection of tuples indexed by $\mathcal{T}$. We write $\trianglelefteq$ for the tree partial order and $<_{lex}$ for the lexicographic order on $\mathcal{T}$. For a node $\eta \in \mathcal{T}$, write $a_{\trianglelefteq\eta}$ for the sequence $\langle a_\nu : \nu \trianglelefteq \eta\rangle$ (consider it as a sequence by ordering lexicographically), and likewise $a_{\triangleleft\eta}$ for $\langle a_\nu : \nu \triangleleft \eta\rangle$. We use the notation $a_{\trianglerighteq\eta}$ and $a_{\triangleright\eta}$ similarly. If the tree $\mathcal{T}$ is a contained in $2^{<\kappa}$ or $\omega^{<\kappa}$, we write 0^α to denote the element of the tree of length α consisting of all zeros. Throughout the paper, T denotes a complete theory and $\mathbb{M} \models T$ is a monster model of T.

Definition 2.1. [DS04, Definition 2.2] The formula $\varphi(x;y)$ has SOP$_1$ if there is a collection of tuples $(a_\eta)_{\eta\in 2^{<\omega}}$ so that

- For all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent.
- For all $\eta \in 2^{<\omega}$, if $\nu \trianglerighteq \eta \frown 0$, then $\{\varphi(x; a_\nu), \varphi(x; a_{\eta\frown 1})\}$ is inconsistent.

We say T is SOP$_1$ if some formula has SOP$_1$ modulo T. T is NSOP$_1$ otherwise.

The following lemma closely follows the proof of Lemma 5.2 in [CR16], but with a strengthening which allows us to relax the 2-inconstency in the definition of SOP$_1$ to a version with k-inconsistency.

{fancy}

Lemma 2.2. *Suppose $(c_{i,j})_{i<\omega, j<2}$ is an array where $c_{i,j} = (d_{ij}, e_{ij})$ for all i, j and $\chi_1(x;y)$ and $\chi_2(x;z)$ are formulas over C. Write $\psi(x;y,z)$ for $\chi_1(x;y) \wedge \chi_2(x;z)$ and suppose*

(1) For all $i < \omega$, $c_{i,0} \equiv_{Cc_{<i,0}e_{<i,1}} c_{i,1}$;
(2) $\{\psi(x;c_{i,0}) : i < \omega\}$ is consistent;
(3) $j \le i \implies \{\chi_1(x;d_{i,0}), \chi_2(x;e_{j,1})\}$ is inconsistent,

then T is SOP$_1$.

Proof. By adding constants, we may assume $C = \emptyset$. For each n, define a subtree T_n by

$$T_n = \{\eta \frown 0^\alpha : \eta \in 2^{\le n}, \alpha < \omega\} \cup \{\eta \frown 0^\alpha \frown 1 : \eta \in 2^{\le n}, \alpha < \omega\}.$$

Let $P(T_n) \subseteq 2^\omega$ be the set of infinite paths of T_n. Concretely,

$$P(T_n) = \{\eta \frown 0^\omega : \eta \in 2^{\le n}\}$$

As a first step, we will build by induction an ascending sequence of tuples $(l_\eta, r_\eta)_{\eta\in T_n}$, where $l_\eta = (d_\eta, e_\eta)$ and $r_\eta = (f_\eta, g_\eta)$ so that

(1) If $\eta \in P(T_n)$, $(l_{\eta|\alpha}, g_{\eta|\alpha})_{\alpha<\omega} \equiv (c_{\alpha,0}, e_{\alpha,1})_{\alpha<\omega}$.

(2) If $\eta \frown 1 \in T_n$ then $r_{\eta \frown 0} = l_{\eta \frown 1}$.
(3) If $\eta \in 2^{\leq n}$ then $(l_{\eta \frown 0}, r_{\eta \frown 0}) \equiv_{l_{\trianglelefteq \eta} g_{\trianglelefteq \eta}} (l_{\eta \frown 1}, r_{\eta \frown 1})$.

For the $n = 0$ case, define $l_{0^\alpha} = (d_{0^\alpha}, e_{0^\alpha}) = c_{\alpha,0}$, $r_{0^\alpha} = (f_{0^\alpha}, g_{0^\alpha}) = c_{\alpha,1}$ and $l_{0^\alpha \frown 1} = r_{0^\alpha \frown 0}$ for all $\alpha < \omega$. For each $\alpha < \omega$, we can choose $\sigma_\alpha \in \text{Aut}(\mathbb{M}/c_{<\alpha,0} e_{<\alpha,1})$ such that $\sigma_\alpha(c_{\alpha,0}) = c_{\alpha,1}$. Let $r_{0^\alpha \frown 1} = \sigma_{\alpha+1}(c_{\alpha+1,1}) = \sigma_{\alpha+1}(r_{0^\alpha \frown 0})$. This defines $(l_\eta, r_\eta)_{\eta \in T_0}$ satisfying (1)-(3).

Now by induction suppose $(l_\eta, r_\eta)_{\eta \in T_n}$ has been defined. Suppose $\eta \in P(T_{n+1}) \setminus P(T_n)$. Then there is $\nu \in 2^{\leq n}$ so that $\eta = \nu \frown 1 \frown 0^\omega$. Then $\nu \frown 1 \in T_n$ and, by induction,

$$(l_{\nu \frown 0}, r_{\nu \frown 0}) \equiv_{l_{\trianglelefteq \nu} g_{\trianglelefteq \nu}} (l_{\nu \frown 1}, r_{\nu \frown 1})$$

and $r_{\nu \frown 0} = l_{\nu \frown 1}$. Choose an automorphism $\sigma \in \text{Aut}(\mathbb{M}/l_{\trianglelefteq \nu} g_{\trianglelefteq \nu})$ so that $\sigma(l_{\nu \frown 0}) = l_{\nu \frown 1}$. Then put $r_{\nu \frown 1} = \sigma(r_{\nu \frown 0})$. Then define $(l_{\nu \frown 1 \frown 0^\alpha}, r_{\nu \frown 1 \frown 0^\alpha}) = \sigma(l_{\nu \frown 0 \frown 0^\alpha}, r_{\nu \frown 0 \frown 0^\alpha})$ for all $\alpha < \omega$. This completes the construction of $(l_\eta, r_\eta)_{\eta \in T_{n+1}}$. We obtain $(l_\eta, r_\eta)_{\eta \in 2^{<\omega}}$ as the union over all n of $(l_\eta, r_\eta)_{\eta \in T_n}$.

Now we check that with respect to the parameters $(l_\eta)_{\eta \in 2^{<\omega}}$, ψ witnesses SOP$_1$. Fix any path $\eta \in 2^\omega$, we have to check that $\{\psi(x; l_{\eta|\alpha}) : \alpha < \omega\}$ is consistent. But given any n, $l_{\trianglelefteq (\eta|n)} \subset T_n$ and by (1), $l_{\trianglelefteq (\eta|n)} \equiv (c_{\alpha,0})_{\alpha \leq n}$ hence $\{\psi(x; l_{\eta|\alpha}) : \alpha \leq n\}$ is consistent, as $\{\psi(x; c_{\alpha,0}) : \alpha \leq n\}$ is consistent, by hypothesis. Then $\{\psi(x; l_{\eta|\alpha}) : \alpha < \omega\}$ is consistent by compactness.

Now fix $\eta \perp \nu \in 2^{<\omega}$ so that $(\eta \wedge \nu) \frown 0 \trianglelefteq \eta$ and $(\eta \wedge \nu) \frown 1 = \nu$. We must check $\{\psi(x; l_\eta), \psi(x; l_\nu)\}$ is inconsistent. By definition of ψ, it is enough to show $\{\chi_1(x; d_\eta), \chi_2(x; e_\nu)\}$ is inconsistent. As $\nu = (\eta \wedge \nu) \frown 1$, we know that $l_\nu = l_{(\eta \wedge \nu) \frown 1} = r_{(\eta \wedge \nu) \frown 0}$ by (2). Let $\xi = (\eta \wedge \nu) \frown 0$. Then $\xi \trianglelefteq \eta$ and $e_\nu = g_\xi$ so it suffices to show $\{\chi_1(x; e_\eta), \chi_2(x; g_\xi)\}$ is inconsistent. Let $n = l(\eta)$ and $m = l(\xi)$. Then $m \leq n$ and by (1), we have $(l_\eta, g_\xi) \equiv_C (c_{n,0}, c_{m,1})$. By hypothesis, this implies $\{\chi_1(x; d_\eta), \chi_2(x; g_\xi)\}$ is inconsistent, so we finish. □

{karyversion}

Lemma 2.3. *Suppose $\varphi(x; y)$ is a formula, k is a natural number, and $(\overline{c}_i)_{i \in I}$ is an infinite indiscernible sequence with $\overline{c}_i = (c_{i,0}, c_{i,1})$ satisfying:*

(1) For all $i \in I$, $c_{i,0} \equiv_{\overline{c}_{<i}} c_{i,1}$
(2) $\{\varphi(x; c_{i,0}) : i \in I\}$ is consistent
(3) $\{\varphi(x; c_{i,1}) : i \in I\}$ is k-inconsistent,

then T has SOP$_1$.

Proof. By compactness, it suffices to prove this when $I = \mathbb{Q}$ – so suppose $(c_{i,0}, c_{i,1})_{i \in \mathbb{Q}}$ is an indiscernible sequence with $c_{i,0} \equiv_{\overline{c}_{<i}} c_{i,1}$, $\{\varphi(x; c_{i,0}) : i \in \mathbb{Q}\}$ is consistent, and $\{\varphi(x; c_{i,1}) : i \in \mathbb{Q}\}$ is k-inconsistent.

For integers $l < l'$, define a partial type $\Gamma_{l,l'}(x)$ by

$$\{\varphi(x; c_{i,0}) : i \in (l+m, l+m+1), m \in \omega, m < l'-l\} \cup \{\varphi(x; c_{l+m,1}) : m < l'-l, m \in \omega\}.$$

Let $\Gamma_{l,l}(x) = \emptyset$. Let n be maximal so that $\Gamma_{0,n}(x)$ is consistent. Note that if $\Gamma_{l,l'}(x)$ is consistent then $\Gamma_{l+z,l'+z}(x)$ is consistent for any integer z by indiscernibility of the sequence $(\overline{c}_i)_{i \in \mathbb{Q}}$. Let $n \in \omega$ be maximal so that $\Gamma_{0,n}(x)$ is consistent. Note that $\Gamma_{0,0}(x)$ is consistent, as it is the empty partial type and we have

$$\Gamma_{0,k}(x) \vdash \{\varphi(x; c_{i,1}) : i \in \omega, i < k\},$$

which is inconsistent, so $0 \leq n < k$. So now we know $\Gamma_{-n,0}(x)$ is consistent and $\Gamma_{-n,1}(x) = \Gamma_{-n,0}(x) \cup \Gamma_{0,1}(x)$ is inconsistent. By indiscernibility and compactness, we may fix some integer $N > 0$ so that

$$\Gamma_{-n,0}(x) \cup \{\varphi(x; c_{0,1})\} \cup \{\varphi(x; c_{\frac{j+1}{N},0}) : j \in \omega, j < N-1\}$$

is inconsistent. Now choose $\Delta(x) \subseteq \Gamma_{-n,0}(x)$ finite so that

$$\Delta(x) \cup \{\varphi(x; c_{0,1})\} \cup \{\varphi(x; c_{\frac{j+1}{N},0}) : j \in \omega, j < N-1\}$$

is inconsistent. Let z indicate the tuple of variables $(y_0, \ldots, y_{N-2})$ and let $\chi(x; z)$ be the formula $\chi(x; z) = \bigwedge_{i<N} \varphi(x; y_i) \wedge \bigwedge \Delta(x)$. Let $(a_{i,j})_{i<\omega, j<2}$ be defined as follows:

$$a_{i,0} = (c_{i,0}; d_{i,0}) = (c_{i,0}; c_{i+\frac{1}{N},0}, \ldots, c_{i+\frac{N-1}{N},0}).$$

Now choose $d_{i,1}$ so that $c_{i,0}d_{i,0} \equiv_{\bar{c}_{<i}} c_{i,1}d_{i,1}$ – this is possible as $c_{i,0} \equiv_{\bar{c}_{<i}} c_{i,1}$. Then we put $a_{i,1} = (c_{i,1}, d_{i,1})$. Let $\psi(x; yz) = \varphi(x; y) \wedge \chi(x; z)$.

To conclude, we have to establish the following:

Claim: The array $(a_{i,j})_{i<\omega,j<2}$ and the formulas $\varphi(x; y)$, $\chi(x; z)$ satisfy

(1) $a_{i,0} \equiv_{a_{<i,0}, c_{<i,1}} a_{i,1}$
(2) $\{\psi(x; a_{i,0}) : i < \omega\}$ is consistent
(3) If $l \leq l'$ then $\{\varphi(x; c_{l,1}), \chi(x; d_{l',0})\}$ is inconsistent.

Proof of claim: (1) follows from the fact that $a_{i,0} \equiv_{\bar{c}_{<i}} a_{i,1}$ and both $a_{<i,0}$ and $c_{<i,1}$ are enumerated in $\bar{c}_{<i}$. Note that $\Gamma_{-n,0}(x)$ is consistent so, by indiscernibility,

$$\Gamma_{-n,0}(x) \cup \{\varphi(x; c_{i,0}) : i \in [0, \infty) \cap \mathbb{Q}\}$$

is consistent, which establishes (2). Finally, if $l \leq l'$, then $\{\varphi(x; c_{l,1}), \chi(x; d_{l',0})\}$ implies

$$\{\varphi(x; c_{l,1})\} \cup \{\varphi(x; c_{l'+\frac{j+1}{N}}) : j \in \omega, j < N-1\} \cup \Delta(x).$$

By indiscernibility of $(\bar{c}_i)_{i\in\mathbb{Q}}$ and the fact that $l \leq l'$, this set is consistent if and only if

$$\{\varphi(x; c_{0,1})\} \cup \{\varphi(x; c_{\frac{j+1}{N},0}) : j \in \omega, j < N-1\} \cup \Delta(x)$$

is consistent. As this latter set is inconsistent, this shows (3), which proves the claim. The lemma now follows by Lemma 2.2. □

Finally, we note that the criterion for SOP_1 from Lemma 2.3 is an equivalence. This was implicit in [CR16], at least in its 2-inconsistent version, but we think that the property described by Lemma 2.3 is, in most cases, the more fruitful way of thinking about SOP_1 and therefore worth making explicit.

{arrayequivalent}

Proposition 2.4. *The following are equivalent:*

(1) φ has SOP_1
(2) There is an array $(c_{i,j})_{i<\omega,j<2}$ so that
  (a) $c_{i,0} \equiv_{c_{<i}} c_{i,1}$ for all $i < \omega$
  (b) $\varphi(x; c_{i,0}) : i < \omega\}$ is consistent
  (c) $\{\varphi(x; c_{i,1}) : i < \omega\}$ is 2-inconsistent.
(3) There is an array $(c_{i,j})_{i<\omega,j<2}$ so that
  (a) $c_{i,0} \equiv_{c_{<i}} c_{i,1}$ for all $i < \omega$
  (b) $\varphi(x; c_{i,0}) : i < \omega\}$ is consistent
  (c) $\{\varphi(x; c_{i,1}) : i < \omega\}$ is k-inconsistent for some k.

Proof. (3) $\Longrightarrow$ (1) is Lemma 2.3.

(1) $\Longrightarrow$ (2). This follows from the proof of the proof of [CR16, Proposition 5.6]

(2) $\Longrightarrow$ (3) is obvious. □

3. Kim-dividing

3.1. Averages and Invariant Types.

Definition 3.1. A global type $q \in S(\mathbb{M})$ is called *A-invariant* if $b \equiv_A b'$ implies $\varphi(x;b) \in q$ if and only if $\varphi(x;b') \in q$. A global type q is *invariant* if there is some small set A such that q is A-invariant. If $q(x)$ and $r(y)$ are A-invariant global types, then the type $(q \otimes r)(x,y)$ is defined to be $\text{tp}(a,b/\mathbb{M})$ for any $b \models r$ and $a \models q|_{\mathbb{M}b}$. We define $q^{\otimes n}$ by induction: $q^{\otimes 1} = q$ and $q^{\otimes n+1} = q^{\otimes n} \otimes q$.

Fact 3.2. [Sim15, Chapter 2] Given a global A-invariant type q and positive integer n, $q^{\otimes n}$ is a well-defined A-invariant global type. If $N \supset A$ is an $|A|^+$-saturated model and $p \in S(N)$ satisfies $\varphi(x;b) \in p \iff \varphi(x;b') \in p$ whenever $b, b' \in N$ and $b \equiv_A b'$, then p extends uniquely to a global A-invariant type. {tensor}

Definition 3.3. Suppose q is an A-invariant global type and I is a linearly ordered set. By a *Morley sequence in q over A of order type I*, we mean a sequence $(b_\alpha)_{\alpha \in I}$ such that for each $\alpha \in I$, $b_\alpha \models q|_{A\overline{b}_{<\alpha}}$ where $\overline{b}_{<\alpha} = (b_\beta)_{\beta<\alpha}$. Given a linear order I, we will write $q^{\otimes I} = q^{\otimes I}(x_\alpha : \alpha \in I)$ for the A-invariant global type so that if $\overline{b} \models q^{\otimes I}$ then $b_\alpha \models q|_{\mathbb{M}\overline{b}_{<\alpha}}$ for all $\alpha \in I$.

The above definition of $q^{\otimes I}$ generalizes the finite tensor product $q^{\otimes n}$ – given any global A-invariant type q and linearly ordered set I, one may easily show that $q^{\otimes I}$ exists and is A-invariant, by Fact 3.2 and compactness.

Definition 3.4. Let $I \subseteq \mathbb{M}^n$ be a sequence of tuples, $A \subseteq \mathbb{M}$ a set, and $\mathcal{D}$ an ultrafilter over I. We define the *average type of $\mathcal{D}$ over A* to be the type defined by

$$\text{Av}(\mathcal{D}, A) = \{\varphi(x;a) : a \in A \text{ and } \{b \in I : \mathbb{M} \models \varphi(b;a)\} \in \mathcal{D}\}.$$

Fact 3.5. [She90, Lemma 4.1] Let $I \subseteq \mathbb{M}^n$ be a collection of tuples and $\mathcal{D}$ an ultrafilter on I. {average}

(1) For every set C, $\text{Av}(\mathcal{D}, C)$ is a complete type over C.
(2) The global type $\text{Av}(\mathcal{D}, \mathbb{M})$ is I-invariant.
(3) For any model $M \models T$, if $p \in S^n(M)$, there is some ultrafilter $\mathcal{E}$ on M^n so that $p = \text{Av}(\mathcal{E}, M)$.

One important consequence Fact 3.5 for us is that every type over a model M extends to a global M-invariant type: given $p \in S(M)$, one chooses an ultrafilter $\mathcal{D}$ so that $\text{Av}(\mathcal{D}, M) = p$. Then $\text{Av}(\mathcal{D}, \mathbb{M})$ is a global type extending p which is M-invariant. In the arguments below, it will often be convenient to produce global invariant types through a particular choice of ultrafilter.

Definition 3.6. Suppose $M \models T$ and $\overline{a} = (a_i)_{i<\omega}$ is an M-indiscernible sequence. A global M-invariant type $q \supseteq \text{tp}(\overline{a}/M)$ is called an *indiscernible type* if whenenver $\overline{a}' \models q$, $\overline{a}'$ is $\mathbb{M}$-indiscernible.

The following two lemma are essentially [Adl14, Lemma 8]. We include a proof for completeness.

Lemma 3.7. *If $\overline{a} = (a_i)_{i<\omega}$ is an M-indiscernible sequence, there is an indiscernible global M-invariant type $q \supseteq tp(\overline{a}/M)$.*

Proof. Let N be an $|M|^+$-saturated elementary extension of M of size κ and let $\lambda = \beth_\omega((2^\kappa)^+)$. By compactness, we may stretch the given indiscernible sequence to $\overline{a}' = (a'_i)_{i<\lambda}$. Let $r \supseteq \text{tp}(\overline{a}'/M)$ be an arbitrary global M-invariant type extending $\text{tp}(\overline{a}'/M)$. Let $\overline{d} = (d_i)_{i<\lambda} \models r|_N$. Use Erdős-Rado to extract from $\overline{d}$ an N-indiscernible sequence $(b_i)_{i<\omega}$. Clearly $\text{tp}(\overline{b}/N)$ extends $\text{tp}(\overline{a}/M)$. It is also M-invariant: if not, there are $n \equiv_M n'$ in N, an increasing k-tuple $\overline{i}$ from ω, and a formula φ so that

$$\models \varphi(b_{\overline{i}}; n) \leftrightarrow \neg\varphi(b_{\overline{i}}, n').$$

Then there is an increasing k-tuple $\overline{j}$ so that

$$\models \varphi(d_{\overline{j}}; n) \leftrightarrow \neg\varphi(d_{\overline{j}}; n'),$$

since the sequence $\overline{b}$ is based on $\overline{d}$. This contradicts the fact that $\overline{d}$ realizes an M-invariant type over N. By Fact 3.2, the type $\text{tp}(\overline{d}/N)$ determines a unique M-invariant extension to $\mathbb{M}$. Call it q. Then q is an indiscernible type. □

{pathtype}

Lemma 3.8. *Suppose $M \models T$, $\overline{a} = (a_i)_{i<\omega}$ is an M-indiscernible sequence, and $q \supseteq tp(\overline{a}/M)$ is a global M-invariant indiscernible type. Let $(\overline{a}_i)_{i<\omega} \models q^{\otimes\omega}|_M$ with $\overline{a}_0 = \overline{a}$, where $\overline{a}_i = (a_{i,j})_{j<\omega}$. Then $(\overline{a}_i)_{i<\omega}$ is a mutually indiscernible array over M.*

Proof. We prove by induction on n that $(\overline{a}_i)_{i\leq n}$ is mututally indiscernible over M. For $n = 1$, there's nothing to prove. Suppose it's been shown for n and consider $(a_i)_{i\leq n+1}$. As q is an indiscernible type, $\overline{a}_{n+1}$ is $M\overline{a}_{\leq n}$-indiscernible. For $i \leq n$, we know, by induction, that $\overline{a}_i$ is $M\overline{a}_{<i}\overline{a}_{i+1}\ldots\overline{a}_n$-indiscernible. As $\overline{a}_{n+1} \models q|_{M\overline{a}_{\leq n}}$, this entails $\overline{a}_i$ is indiscernible over $M\overline{a}_{<i}\overline{a}_{i+1}\ldots\overline{a}_{n+1}$, which completes the induction. □

3.2. **Kim-dividing.** In this subsection, we define Kim-dividing and Kim-forking, the fundamental notions explored in this paper. To start, we will need the definition of q-dividing, introduced by Hrushovski in [Hru12, Section 2.1]:

Definition 3.9. Suppose $q(y)$ is an A-invariant global type. The formula $\varphi(x; y)$ q-divides over A if for some (equivalently, any) Morley sequence $\langle b_i : i < \omega\rangle$ in q over A, $\{\varphi(x; b_i) : i < \omega\}$ is inconsistent.

We note that we will consistently use the letters p, q, r to refer to types, n, m, k, l to refer to numbers. In this way, no confusion between q-dividing and the more familiar k-dividing will arise.

The related notion of a *higher formula* was introduced by Malliaris and Shelah in [MS15] on the way to a new characterization of NTP_1 theories:

Definition 3.10. [MS15, Definition 8.6] A *higher formula* is a triple $(\varphi, A, \mathcal{D})$ where $\varphi = \varphi(x; y)$ is a formula, A is a set of parameters, and $\mathcal{D}$ is an ultrafilter on $A^{l(y)}$ so that, if $q = \text{Av}(\mathcal{D}, \mathbb{M})$ and $\langle b_i : i < \omega\rangle \models q^{\otimes\omega}|_A$ then $\{\varphi(x; b_i) : i < \omega\}$ is consistent.

We can rephrase the above definition as: $(\varphi, A, \mathcal{D})$ is a higher formula if, setting $q = \text{Av}(\mathcal{D}, \mathbb{M})$, $\varphi(x; y)$ does not q-divide over A.

Definition 3.11. We say that a formula $\varphi(x; b)$ *Kim-divides* over A if there is some A-invariant global type $q \supseteq \text{tp}(b/A)$ so that $\varphi(x; y)$ q-divides. The formula $\varphi(x; b)$ *Kim-forks* over A if $\varphi(x; b) \vdash \bigvee_{i<k} \psi_i(x; c^i)$ and each $\psi_i(x; c^i)$ Kim-divides

over A. A type Kim-forks if it implies a formula which does. If $\mathrm{tp}(a/Ab)$ does not Kim-fork over A, we write $a \mathop{\smile\!\!\!\!\!\mid}^K_A b$.

We call this notion *Kim*-dividing to make explicit the fact that this definition was inspired by a suggestion of Kim in his 2009 BIRS talk [Kim09], where he proposed an independence relation based on instances of dividing that are witnessed by every appropriate Morley sequence. A rough connection between Kim's notion and ours is provided by Theorem 3.14 below, which shows that, in an NSOP_1 theory, dividing with respect some invariant Morley sequence is equivalent to dividing with respect to all. An even tighter connection is established by Theorem 6.6, which shows that we can drop the assumption that the Morley sequences are generated by an invariant type. (We note that for technical reasons our notion is still different from Kim's – the proposal of [Kim09] forces a kind of base monotonicity and we do not).

In general, we only know that a type over A has a global A-invariant extension when A is a model. Thus, when working with Kim-independence below, we will restrict ourselves almost entirely to the case where the base is a model.

The next two propositions explain how the notions of higher formula and q-dividing interact with SOP_1.

{higher}

Proposition 3.12. *Suppose T has SOP_1. Then there is a model $M \models T$, a formula $\varphi(x;b)$, and ultrafilters $\mathcal{D}_0, \mathcal{D}_1$ on M with*

$$Av(\mathcal{D}_0, M) = Av(\mathcal{D}_1, M) = tp(b/M),$$

so that $(\varphi, M, \mathcal{D}_0)$ is higher but $(\varphi, M, \mathcal{D}_1)$ is not higher.

Proof. Fix a Skolemization T^{Sk} of M. As T has SOP_1, there is, by Proposition 2.4, a formula $\varphi(x;y)$ and an array $(c_{i,j})_{i<\omega+1,j<2}$ such that

(1) $(\overline{c}_i)_{i<\omega+1}$ is an indiscernible sequence (with respect to the Skolemized language)
(2) $c_{\omega,0} \equiv^{L^{Sk}}_{\overline{c}_{<\omega}} c_{\omega,1}$.
(3) $\{\varphi(x;c_{i,0}) : i < \omega+1\}$ is consistent.
(4) If $i<j$, then $\{\varphi(x;c_{i,1}), \varphi(x;c_{j,0})\}$ is inconsistent.

Put $M = \mathrm{Sk}(\overline{c}_{<\omega})$. For $j = 0,1$, let $\mathcal{D}_j$ be any non-principal ultrafilter on M, concentrating on $\langle c_{i,j} : i < \omega \rangle$ and set $q_j = \mathrm{Av}_L(\mathcal{D}_j, \mathbb{M})$ for $j = 0,1$. Note that $q_0|_M = \mathrm{tp}_L(c_{\omega,0}/M) = \mathrm{tp}_L(c_{\omega,1}/M) = \mathrm{q}_1|_M$ by (2). By (3), $\varphi(x;y)$ does not q_0-divide, hence $(\varphi, M, \mathcal{D}_0)$ is higher. However, by (4) and indiscernibility, $\{\varphi(x;c_{1,j}) : j<\omega\}$ is 2-inconsistent hence $\varphi(x;y)$ q_1-divides, so $(\varphi, M, \mathcal{D}_1)$ is not higher. □

{qdiv}

Proposition 3.13. *Suppose A is a set of parameters and $\varphi(x;b)$ is a formula which q-divides over A for some global A-invariant type $q \supseteq tp(b/A)$. If there is some global A-invariant $r \supseteq tp(b/A)$ such that $\varphi(x;y)$ does not r-divide, then T has SOP_1.*

Proof. As $\varphi(x;y)$ q-divides over A, there is k so that instances of $\varphi(x;y)$ instantiated on a Morley sequence of q are k-inconsistent.

Let $(c_{i,1}, c_{i,0})_{i\in\mathbb{Z}} \models (q \otimes r)^{\otimes \mathbb{Z}}|_M$. We have to check that the sequence satisfies the following properties:

(1) $\{\varphi(x;c_{i,0}) : i \in \mathbb{Z}\}$ is consistent
(2) $\{\varphi(x;c_{i,1}) : i \in \mathbb{Z}\}$ is k-inconsistent
(3) $c_{i,0} \equiv_{\overline{c}_{>i}} c_{i,1}$ for all $i \in \mathbb{Z}$.

Note that $(c_{i,0})_{i\in\mathbb{Z}} \models r^{\otimes\mathbb{Z}}|_M$ so (1) follows from our assumption that $\varphi(x;y)$ does not r-divide. Likewise, $(c_{i,1})_{i\in\mathbb{Z}} \models q^{\otimes\mathbb{Z}}|_M$ so (2) follows from the fact that $\varphi(x,y)$ q-divides. Finally, for any $i \in \mathbb{Z}$, we have $\overline{c}_{>i}$ realizes a global M-invariant type over $Mc_{i,0}c_{i,1}$. Hence (3) follows from the fact that $c_{i,0} \equiv_M c_{i,1}$. □

{kimslemmaforindk}

Theorem 3.14. *The following are equivalent for the complete theory T:*

1. *T is NSOP$_1$*
2. *Ultrafilter independence of higher formulas: for every model $M \models T$, and ultrafilters $\mathcal{D}$ and $\mathcal{E}$ on M with $Av(\mathcal{D},M) = Av(\mathcal{E},M)$, $(\varphi,M,\mathcal{D})$ is higher if and only if $(\varphi,M,\mathcal{E})$ is higher*
3. *Kim's lemma for Kim-dividing: For every model $M \models T$ and $\varphi(x;b)$, if $\varphi(x;y)$ q-divides for some global M-invariant $q \supseteq tp(b/M)$, then $\varphi(x;y)$ q-divides for* every *global M-invariant $q \supseteq tp(b/M)$.*

Proof. (1) $\Longrightarrow$ (3) is the contrapositive of Proposition 3.13.

(2) $\Longrightarrow$ (1) is the contrapositive of Proposition 3.12.

(3) $\Longrightarrow$ (2): Immediate, since every type finitely satisfiable in M is M-invariant. □

Remark 3.15. Note that the proof gives a bit more: if T is NSOP$_1$, (2) is true over arbitrary sets and (3) is true over an arbitrary set A provided $\text{tp}(b/A)$ extends to a global A-invariant type.

3.3. **The basic properties of Kim-independence.** Theorem 3.14, a kind of Kim's lemma for Kim-dividing, already gives a powerful tool for proving that in NSOP$_1$ theories Kim-independence enjoys many of the properties known to hold for non-forking independence in simple theories.

We will frequently use the following easy observation. The proof is exactly as in the case of dividing. See, e.g., [GIL02, Lemma 1.5] or [She80, Lemma 1.4].

{basiccharacterization}

Lemma 3.16. *(Basic Characterization of Kim-dividing) Suppose T is an arbitrary complete theory. The following are equivalent:*

1. *$tp(a/Ab)$ does not Kim-divide over A.*
2. *For any global A-invariant $q \supseteq tp(b/A)$ and $I = \langle b_i : i < \omega\rangle \models q^{\otimes\omega}|_A$ with $b_0 = b$, there is $a' \equiv_{Ab} a$ such that I is Aa'-indiscernible.*
3. *For any global A-invariant $q \supseteq tp(b/A)$ and $I = \langle b_i : i < \omega\rangle \models q^{\otimes\omega}|_A$ with $b_0 = b$, there is $I' \equiv_{Ab} I$ such that I' is Aa-indiscernible.*

Note that in an NSOP$_1$ theory, by Kim's Lemma for Kim-dividing, we could have replaced (2) by: *there is* a global A-invariant $q \supseteq \text{tp}(b/A)$ and $I = \langle b_i : i < \omega\rangle \models q^{\otimes\omega}|_A$ with $b_0 = b$, so that for some $a' \equiv_{Ab} a$ such that I is Aa'-indiscernible (and similarly for (3)).

The following proposition is proved by the same argument one uses to prove forking = dividing via Kim's lemma, as in [GIL02, Theorem 2.5] or [CK12, Corollary 3.16]

{kforkingequalskdividing}

Proposition 3.17. *(Kim-forking = Kim-dividng) Suppose T is NSOP$_1$. If $M \models T$, if $\varphi(x;b)$ Kim-forks over M then $\varphi(x;b)$ Kim-divides over M.*

Proof. Suppose $\varphi(x;b) \vdash \bigvee_{j<k}\psi_j(x;c^j)$ where each $\psi_i(x;c^i)$ K-divides over M. Fix an ultrafilter $\mathcal{D}$ on M so that $(b,c^0,\ldots,c^{k-1}) \models \text{Av}(\mathcal{D},M)$. Let $(b_i,c_i^0,\ldots,c_i^{k-1})_{i<\omega}$

be a Morley sequence in $\text{Av}(\mathcal{D}, \mathbb{M})$. Then $(b_i)_{i<\omega}$ is an M-invariant Morley sequence. We must show $\{\varphi(x; b_i) : i < \omega\}$ is inconsistent. Suppose not – let $a \models \{\varphi(x; b_i) : i < \omega\}$. We have $\varphi(x; b_i) \vdash \bigvee_{j<k} \psi_j(x; c_i^j)$ so for each $i < \omega$, there is $j(i) < k$ so that $\models \psi_{j(i)}(a; c_i^{j(i)})$. By the pigeonhole principle, there is $j_* < k$ so that $X = \{i < \omega : j(i) = j_*\}$ is infinite. Then $(c_i^{j_*})_{i\in X}$ is an M-invariant Morley sequence in $\text{tp}(c^{j_*}/M)$. As T is NSOP$_1$, Kim-dividing over M is witnessed by any M-invariant Morley sequence so $\{\psi_{j_*}(x; c_i^{j_*}) : i \in X\}$ is inconsistent. But $a \models \{\psi_{j_*}(x; c_i^{j_*}) : i \in X\}$, a contradiction. □

{extension}

Proposition 3.18. *(Extension over Models) Suppose M is a model, and $a \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K_M b$. Then for any c, there is $a' \equiv_{Mb} a$ so that $a' \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K_M bc$.*

Proof. This is the usual proof, using Proposition 3.17. Let $p(x; b) = \text{tp}(a/Mb)$. We claim that the following set of formulas is consistent:

$$p(x; b) \cup \{\neg\psi(x; b, c) : \psi(x; b, c) \in L(Mbc) \text{ and } \psi(x; b, c) \text{ Kim-divides over } M\}.$$

If this set of formulas is not consistent, then by compactness,

$$p(x; b) \vdash \bigvee_{i<k} \psi_i(x; b, c_i),$$

where each $\psi_i(x; b, c_i)$ Kim-divides over M. It follows that $\text{tp}(a/Mb)$ Kim-forks over M, so it Kim-divides over M by Proposition 3.17, a contradiction. So this set is consistent and we may choose a realization a'. Then $a' \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K_M bc$ and $a' \equiv_{Mb} a$. □

{chaincondition}

Proposition 3.19. *(Chain Condition for Invariant Morley Sequences) Suppose T is NSOP$_1$ and $M \models T$. If $a \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K_M b$ and $q \supseteq tp(b/M)$ is a global M-invariant type, then for any $I = \langle b_i : i < \omega\rangle \models q^{\otimes\omega}|_M$ with $b = b_0$, there is $a' \equiv_{Mb} a$ so that $a' \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}_M I$ and I is Ma'-indiscernible.*

Proof. By the basic characterization of Kim-dividing, Lemma 3.16, given $a \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K_M b$, $q \supseteq \text{tp}(b/M)$ a global M-invariant type, and $I = \langle b_i : i < \omega\rangle \models q^{\otimes\omega}|_M$ with $b = b_0$, there is $a' \equiv_{Mb} a$ so that I is Ma'-indiscernible. To prove the proposition it suffices to show $a' \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K_M b_{<n}$ for all n. Given $n < \omega$, let $r(x; y_0, \dots, y_{n-1}) = \text{tp}(a'; b_0, \dots, b_{n-1}/M)$. Then $\langle (b_{kn+n-1}, b_{kn+n-2}, \dots, b_{kn}) : k < \omega\rangle \models (q^{\otimes n})^{\otimes\omega}|_M$ and, by indiscernibility,

$$a' \models \bigcup_{k<\omega} r(x; b_{kn+n-1}, b_{kn+n-2}, \dots, b_{kn}).$$

As T is NSOP$_1$, this shows $a' \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K_M b_{<n}$. □

The next section will be dedicated to the proof that $\mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K$ is symmetric in NSOP$_1$ theories. The argument will require more tools, but at this stage we can already observe the converse: even a weak form of symmetry for $\mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K$ will imply that a theory is NSOP$_1$.

{weaksymmetrylemma}

Proposition 3.20. *The following are equivalent:*

(1) T is NSOP$_1$

(2) Weak symmetry: if $M \models T$, then $b \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^i_M a \implies a \mathop{\smile\!\!\!\!\!\!\!\;\;\mid}^K_M b$.

Proof. (1) $\implies$ (2). As $b \mathop{\smile\!\!\!\!\!\!|}^i_M a$, there is a global M-invariant type $r \supseteq \text{tp}(b/Ma)$. We can find a Morley sequence $I = \langle b_i : i < \omega \rangle$ in $q|_{Ma}$ with $b_0 = b$. Then I is Ma-indiscernible, so no formula in $\text{tp}(a/Mb)$ divides with respect to the sequence I. But by Kim's lemma for Kim-dividing, this implies $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$.

(2) $\implies$ (1). Suppose $a \not\mathop{\smile\!\!\!\!\!\!|}^K_M b$. We argued in the previous direction that $b \mathop{\smile\!\!\!\!\!\!|}^i_M a$ entails the existence of a Morley sequence which does not witness that $\text{tp}(a/Mb)$ Kim-divides over M. This is a failure of Kim's lemma for Kim dividing, so T has SOP_1 by Theorem 3.14. □

{symmetrysection}

4. Symmetry

4.1. **Generalized indiscernibles and a class of trees.** For an ordinal α, the language $L_{s,\alpha}$ be $\langle \trianglelefteq, \wedge, <_{lex}, (P_\beta)_{\beta<\alpha} \rangle$. We may view a tree with α levels and $L_{s,\alpha}$-structures by interpreting $\trianglelefteq$ as the tree partial order, $\wedge$ as the binary meet function, $<_{lex}$ as the lexicographic order, and P_β interpreted to define level β. For the rest of the paper, a tree will be understood to be an $L_{s,\alpha}$-structure for some appropriate α. We will sometimes suppress the α and refer instead to L_s, where the number of predicates is understood from context. We define a class of trees $\mathcal{T}_\alpha$ as follows.

Definition 4.1. Suppose α is an ordinal. We define $\mathcal{T}_\alpha$ to be the set of functions f so that

- $\text{dom}(f)$ is an end-segment of α, possibly empty unless α is a limit.
- $\text{ran}(f) \subseteq \omega$.
- finite support: the set $\{\gamma \in \text{dom}(f) : f(\gamma) \neq 0\}$ is finite.

We interpret $\mathcal{T}_\alpha$ as an $L_{s,\alpha}$-structure by defining

- $f \trianglelefteq g$ if and only if $f \subseteq g$. Write $f \perp g$ if $\neg(f \trianglelefteq g)$ and $\neg(g \trianglelefteq f)$.
- $f \wedge g = f|_{[\beta,\alpha)} = g|_{[\beta,\alpha)}$ where $\beta = \min\{\gamma : f|_{[\gamma,\alpha)} = g|_{[\gamma,\alpha)}\}$, if non-empty (note that β will not be a limit, by finite support). Define $f \wedge g$ to be the empty function if this set is empty (note that this cannot occur if α is a limit).
- $f <_{lex} g$ if and only if $f \triangleleft g$ or, $f \perp g$ with $\text{dom}(f \wedge g) = [\gamma + 1, \alpha)$ and $f(\gamma) < g(\gamma)$
- For all $\beta < \alpha$, $P_\beta = \{f \in \mathcal{T}_\alpha : \text{dom}(f) = [\beta, \alpha)\}$. P_α is only defined on $\mathcal{T}_\alpha$ if α is a successor, in which case it only contains the empty function.

It is easy to check that for all $n < \omega$, $\mathcal{T}_n \cong \omega^{<n}$. For α infinite, however, $\mathcal{T}_\alpha$ will be ill-founded. In particular, P_0 names the level at the *top* of the tree, $P_{\beta+1}$ names the level immediately *below* P_β, and so on.

As many arguments in this paper will involve inductive constructions of trees of tuples indexed by $\mathcal{T}_\alpha$, it will be useful to fix notation as follows:

Definition 4.2. Suppose α is an ordinal.

(1) If $w \subseteq \alpha$, the *restriction of $\mathcal{T}_\alpha$ to the set of levels w* is given by

$$\mathcal{T}_\alpha \restriction w = \{\eta \in \mathcal{T}_\alpha : \min(\text{dom}(\eta)) \in w \text{ and } \beta \in \text{dom}(\eta) \setminus w \implies \eta(\beta) = 0\}.$$

(2) If $\eta \in \mathcal{T}_\alpha$, $\text{dom}(\eta) = [\beta + 1, \alpha)$, and $i < \omega$, let $\eta \frown \langle i \rangle$ denote the function $\eta \cup \{(\beta, i)\}$.

(3) If $\alpha < \beta$, we define the map $\iota_{\alpha\beta} : \mathcal{T}_\alpha \to \mathcal{T}_\beta$ by $\iota_{\alpha\beta}(f) = f \cup \{(\gamma, 0) : \gamma \in \beta \setminus \alpha\}$.

(4) If $\beta < \alpha$, then $\zeta_\beta \in \mathcal{T}_\alpha$ denotes the function with $\mathrm{dom}(\zeta_\beta) = [\beta, \alpha)$ and $\zeta_\beta(\gamma) = 0$ for all $\gamma \in [\beta, \alpha)$.

The function $i_{\alpha\beta}$ includes $\mathcal{T}_\alpha$ into $\mathcal{T}_\beta$ by adding zeros to the bottom of every node in $\mathcal{T}_\alpha$. Clearly if $\alpha < \beta < \gamma$, then $\iota_{\alpha\gamma} = \iota_{\beta\gamma} \circ \iota_{\alpha\beta}$. If β is a limit, then $\mathcal{T}_\beta$ is the direct limit of the $\mathcal{T}_\alpha$ for $\alpha < \beta$ along these maps. Visually, to get $\mathcal{T}_{\alpha+1}$ from $\mathcal{T}_\alpha$, one takes countably many copies of $\mathcal{T}_\alpha$ and adds a single root at the bottom.

Definition 4.3. Suppose I is an L'-structure, where L' is some language.

(1) We say $(a_i : i \in I)$ is a set of *I-indexed indiscernibles* if whenever $(s_0, \ldots, s_{n-1})$, $(t_0, \ldots, t_{n-1})$ are tuples from I with

$$\mathrm{qftp}_{L'}(s_0, \ldots, s_{n-1}) = \mathrm{qftp}_{L'}(t_0, \ldots, t_{n-1}),$$

then we have

$$\mathrm{tp}(a_{s_0}, \ldots, a_{s_{n-1}}) = \mathrm{tp}(a_{t_0}, \ldots, a_{t_{n-1}}).$$

(2) In the case that $L' = L_{s,\alpha}$ for some α, we say that an I-indexed indiscernible is *s-indiscernible.* As the only $L_{s,\alpha}$-structures we will consider will be trees, we will often refer I-indexed indiscernibles in this case as *s-indiscernible trees.*

(3) We say that I-indexed indiscernibles have the *modeling property* if, given any $(a_i : i \in I)$ from $\mathbb{M}$, there is an I-indexed indiscernible $(b_i : i \in I)$ in $\mathbb{M}$ *locally based* on $(a_i : i \in I)$ – i.e., given any finite set of formulas Δ from L and a finite tuple $(t_0, \ldots, t_{n-1})$ from I, there is a tuple $(s_0, \ldots, s_{n-1})$ from I so that

$$\mathrm{qftp}_{L'}(t_0, \ldots, t_{n-1}) = \mathrm{qftp}_{L'}(s_0, \ldots, s_{n-1})$$

and also

$$\mathrm{tp}_\Delta(b_{t_0}, \ldots, b_{t_{n-1}}) = \mathrm{tp}_\Delta(a_{s_0}, \ldots, a_{s_{n-1}}).$$

Fact 4.4. [KKS14, Theorem 4.3] Let denote I_s be the $L_{s,\omega}$-structure $(\omega^{<\omega}, \trianglelefteq, <_{lex}, \wedge, (P_\alpha)_{\alpha<\omega})$ with all symbols being given their intended interpretations and each P_α naming the elements of the tree at level α. Then I_s-indexed indiscernibles have the modeling property. {modeling}

Remark 4.5. Note that the tree $\omega^{<\omega}$ is *not* the same tree as $\mathcal{T}_\omega$, which is ill-founded.

Corollary 4.6. *For any α, $\mathcal{T}_\alpha$-indexed indiscernibles have the modeling property.*

Proof. By Fact 4.4 and compactness. □

Definition 4.7. Suppose $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is a tree of tuples, and C is a set of parameters.

(1) We say $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is *spread out over* C if for all $\eta \in \mathcal{T}_\alpha$ with $\mathrm{dom}(\eta) = [\beta + 1, \alpha)$ for some $\beta < \alpha$, there is a global C-invariant type $q_\eta \supseteq \mathrm{tp}(a_{\trianglerighteq\eta\frown 0}/C)$ so that $(a_{\trianglerighteq\eta\frown\langle i\rangle})_{i<\omega}$ is a Morley sequence over C in q_η.

(2) Suppose $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is a tree which is spread out and s-indiscernible over C and for all $w, v \in [\alpha]^{<\omega}$ with $|w| = |v|$,

$$(a_\eta)_{\eta\in\mathcal{T}_\alpha\restriction w} \equiv_C (a_\eta)_{\eta\in\mathcal{T}_\alpha\restriction v}$$

then we say $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is a *Morley tree* over C.

(3) A *tree Morley sequence* over C is a C-indiscernible sequence of the form $(a_{\zeta_\beta})_{\beta<\alpha}$ for some Morley tree $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ over C.

Remark 4.8. If $(a_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over C, then, in order to be spread out over C, it suffices to have global C-invariant types as in (1) for all η identically zero - i.e. those nodes in the tree of the form ζ_β for some $\beta < \alpha$. Note that the condition in (2) forces $(a_{\zeta_\beta})_{\beta<\alpha}$ to be C-indiscernible. Finally, in (3) we speak of $(a_{\zeta_\beta})_{\beta<\alpha}$, the sequence indexed by the all-zeroes path in the tree, simply because this is a convenient choice of a path. In an s-indiscernible tree over C, any two paths will have the same type over C. Hence, (3) may be stated more succinctly as: a tree Morley sequence over C is a path in some Morley tree over C.

{concatenation}

Lemma 4.9. *Suppose $(a_i)_{i<\omega}$ is a tree Morley sequence over C.*

(1) If $a_i = (b_i, c_i)$ for all $i < \omega$, where the b_i's are all initial subtuples of a_i of the same length, then $(b_i)_{i<\omega}$ is a tree Morley sequence over C.
(2) Given $1 \leq n < \omega$, suppose $d_i = (a_{n\cdot i}, a_{n\cdot i+1}, \dots, a_{n\cdot i+n-1})$. Then $(d_i)_{i<\omega}$ is a tree Morley sequence over C.

Proof. (1) is immediate from the definition: s-indiscernibility, spread-outness, and being a Morley tree over C are all preserved under taking subtuples.

(2) Suppose $(a_\eta)_{\eta\in\mathcal{T}_\omega}$ is a Morley tree over C with $a_{\zeta_i} = a_i$. Define a function $j : \mathcal{T}_\omega \to \mathcal{T}_\omega$ so that if $\eta \in \mathcal{T}_\omega$ with $\mathrm{dom}(\eta) = [k,\omega)$, then $\mathrm{dom}(j(\eta)) = [n(k+1),\omega)$ and

$$j(\eta)(l) = \begin{cases} \eta\left(\frac{l}{n} - 1\right) & \text{if } n|l \\ 0 & \text{otherwise} \end{cases}$$

for all $l \in [n(k+1),\omega)$. Define $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ by

$$b_\eta = (a_{j(\eta)}, a_{j(\eta)\frown 0}, \dots, a_{j(\eta)\frown 0^{n-1}}).$$

It is easy to check that this is also an s-indiscernible tree over M (more formally, this construction corresponds to the n-fold elongation of the tree $(a_\eta)_{\eta\in\mathcal{T}_\omega}$ as defined in [CR16] so $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ is s-indiscernible over M by [CR16, Proposition 2.1(1)] there). It is also easy to check that $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ is spread out over M. Finally, the tree $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ is also a Morley tree over M: given $w \in [\omega]^{<\omega}$, let $w' = \{n(k+1)-l : k \in w, l < n\}$. Then if $w, v \in [\omega]^{<\omega}$ and $|w| = |v|$, then $|w'| = |v'|$ so $(a_\eta)_{\eta\in\mathcal{T}_\omega\restriction w'} \equiv_C (a_\eta)_{\eta\in\mathcal{T}_\omega\restriction v'}$ so $(b_\eta)_{\eta\in\mathcal{T}_\omega\restriction w} \equiv_C (b_\eta)_{\eta\in\mathcal{T}_\omega\restriction v}$. It follows that $(b_{\zeta_i})_{i<\omega}$ is a tree Morley sequence over C. As we have

$$b_{\zeta_i} = (a_{\zeta_{n(i+1)}}, a_{\zeta_{n(i+1)}\frown 0}, \dots, a_{\zeta_{n(i+1)\frown 0^{n-1}}}) = (a_{n(i+1)}, a_{n(i+1)+1}, \dots, a_{n(i+1)+n+1}),$$

we deduce that $(d_i)_{i<\omega}$ is a tree Morley sequence over M. □

From the existence of a sufficiently large tree which is spread out and s-indiscernible over M, one can obtain a Morley tree which is based on it. The proof is via a standard Erdős-Rado argument. We follow the argument of [GIL02, Theorem 1.13].

{morleyextraction}

Lemma 4.10. *Suppose $(a_\eta)_{\eta\in\mathcal{T}_\kappa}$ is a tree of tuples, spread out and s-indiscernible over M. If κ is sufficiently large, then there is a Morley tree $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ so that for all $w \in [\omega]^{<\omega}$, there is $v \in [\kappa]^{<\omega}$ so that*

$$(a_\eta)_{\eta\in\mathcal{T}_\kappa\restriction v} \equiv_M (b_\eta)_{\eta\in\mathcal{T}_\omega\restriction w}.$$

Proof. Let $\lambda = 2^{|M|+|T|}$ and set $\kappa = \beth_{\lambda^+}(\lambda)$. Given a tree $(a_\eta)_{\eta\in\mathcal{T}_\kappa}$ s-indiscernible and spread out over M, let

$$\Gamma_n = \{\mathrm{tp}((a_\eta)_{\eta\in\mathcal{T}_\omega\restriction w}/M) : w \in [\kappa]^n\}.$$

By induction on n, we will find a sequence of types $p_n \in \Gamma_n$ so that

$$\Delta(x_\eta : \eta \in \mathcal{T}_\omega) = \bigcup_{n<\omega} \bigcup_{w \in [\omega]^n} p_n(x_\eta : \eta \in \mathcal{T}_\omega \upharpoonright w).$$

is consistent. Construct by induction on n cofinal subsets $F_n \subseteq \lambda^+$ and subsets $X_{\xi,n} \subseteq \kappa$ so that

(1) $F_{n+1} \subseteq F_n$
(2) $|X_{\xi,n}| > \beth_\alpha(\lambda)$ when ξ is the αth element of F_n
(3) If $w \in [X_{\xi,n}]^n$, then $(a_\eta)_{\eta \in \mathcal{T}_\kappa \upharpoonright w} \models p_n$.
(4) $|F_n| = \lambda^+$.

For $n = 0$, we let $F_0 = \lambda^+$ and $X_{\xi,0} = \kappa$ for all $\xi < \lambda^+$. Suppose F_n and $(X_{\xi,n})_{\xi \in F_n}$ have been constructed. Write $F_n = \{\xi_\alpha : \alpha < \lambda^+\}$ where the ξ_α enumerate F_n in increasing order. Then for all $\alpha < \lambda^+$,

$$|X_{\xi_{\alpha+n+1},n}| > \beth_{\alpha+n+1}(\lambda).$$

For a moment, fix $\xi = \xi_{\alpha+n+1}$. Define a coloring on $[X_{\xi,n}]^{n+1}$ by

$$w \mapsto \text{tp}((a_\eta)_{\eta \in \mathcal{T}_\kappa \upharpoonright w}/M).$$

This is a coloring with at most λ many colors so by Erdős-Rado there is a homogeneous subset $X_{\xi,n+1} \subseteq X_{\xi,n}$ with $|X_{\xi,n+1}| > \beth_\alpha(\lambda)$. Let $p_{n+1,\alpha+n+1}$ denote its constant value. By the pigeonhole principle, as the set of possible values is λ and $\{\alpha+n+1 : \alpha < \lambda^+\}$ has size λ^+, there must be some subset $Y \subseteq \{\alpha+n+1 : \alpha < \lambda^+\}$ of cardinality λ^+ so that $\beta, \beta' \in Y$ implies $p_{n+1,\beta} = p_{n+1,\beta'}$. Let $p_{n+1} = p_{n+1,\beta}$ for some/all $\beta \in Y$. Put $F_{n+1} = \{\xi_\beta : \beta \in Y\}$. Then p_{n+1}, F_{n+1}, and $(X_{\xi,n+1})_{\xi \in F_{n+1}}$ clearly satisfy the requirements.

By compactness, this shows that $\Delta(x_\eta : \eta \in \mathcal{T}_\omega)$ is consistent. Let $(b_\eta)_{\eta \in \mathcal{T}_\omega}$ be a realization - now $(b_\eta)_{\eta \in \mathcal{T}_\omega}$ is a Morley tree over M. GIVE DETAILS □

4.2. **The symmetry characterization of NSOP$_1$.** In this subsection, we prove a version of Kim's lemma for tree Morley sequences and use it to prove that Kim-independence is symmetric over models in an NSOP$_1$ theory. Lemma 4.11 is the key step, showing that tree Morley sequences exist under certain assumptions. The method of proof is an inductive construction of a spread out s-indiscernible tree, from which a Morley tree (and hence a tree Morley sequence) can then be extracted. This basic proof-strategy will be repeated several times throughout the paper.

{treeexistence}

Lemma 4.11. *Suppose T is NSOP$_1$, $M \models T$, and $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$. For any ordinal $\alpha \geq 1$, there is a spread out s-indiscernible tree $(c_\eta)_{\eta \in \mathcal{T}_\alpha}$ over M, so that if $\eta \lhd \nu$ and dom$(\nu) = \alpha$, then $c_\eta c_\nu \equiv_M ab$.*

Proof. We will argue by induction on α. For the case $\alpha = 1$, fix $q \supseteq \text{tp}(b/M)$, a global M-invariant type. Let $\langle b_i : i < \omega \rangle \models q^{\otimes \omega}|_M$. As $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$, we may assume this sequence is Ma-indiscernible. Put $c^1_\emptyset = a$ and $c_{\langle i \rangle} = b_i$. It is now easy to check that $(c^1_\eta)_{\eta \in \mathcal{T}_1}$ is a spread out s-indiscernible tree satisfying the requirements.

Suppose for α we've constructed $(c^\beta_\eta)_{\eta \in \mathcal{T}_\beta}$ for $1 \leq \beta \leq \alpha$ such that, if $\gamma < \beta \leq \alpha$ and $\eta \in \mathcal{T}_\gamma$ then $c^\gamma_\eta = c^\beta_{i_{\gamma\beta}(\eta)}$. By spread-outness, we know that $\langle c^\alpha_{\unrhd \langle i \rangle} : i < \omega \rangle$ is an M-invariant Morley sequence which is, by s-indiscernibility over M, $Mc^\alpha_\emptyset$-indiscernible. Therefore, $c^\alpha_\emptyset \mathop{\smile\!\!\!\!\!\!|}^K_M (c^\alpha_{\unrhd \langle i \rangle})_{i<\omega}$. By extension (Proposition 3.18), we may find some $c' \equiv_{M(c^\alpha_{\trianglerighteq \langle i \rangle})_{i<\omega}} c^\alpha_\emptyset$ so that

$$c' \mathop{\smile\!\!\!\!\!\!\;|}^K_M (c^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}.$$

Choose a global M-invariant type $q \supseteq \text{tp}((c^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}/M)$. Let $\langle (c^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \omega \rangle \models q^{\otimes \omega}|_M$ with $c^\alpha_{\eta,0} = c^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$. By the chain condition (Lemma 3.19), we can find $c'' \equiv_{M(c^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}} c'$ so that $c'' \mathop{\smile\!\!\!\!\!\!\;|}^K_M (c^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha, i<\omega}$ and $\langle (c^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \omega \rangle$ is Mc''-indiscernible. Define a new tree $(d_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ by setting $d_\emptyset = c''$ and $d_{i_{\alpha\alpha+1}(\eta)} = c^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$. Then let $(c^{\alpha+1}_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ be a tree s-indiscernible over M locally based on $(d_\eta)_{\eta \in \mathcal{T}_\alpha}$. By an automorphism, we may assume $c^{\alpha+1}_{i_{\alpha\alpha+1}(\eta)} = c^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$. This satisfies our requirements.

Finally, suppose for δ limit we've constructed $(c^\beta_\eta)_{\eta \in \mathcal{T}_\beta}$ for $1 \leq \beta < \delta$ such that, if $\gamma < \beta < \delta$ and $\eta \in \mathcal{T}_\gamma$ then $c^\gamma_\eta = c^\beta_{i_{\gamma\beta}(\eta)}$. If $\eta \in \mathcal{T}_\delta$, then for some $\beta < \delta$, there is $\nu \in \mathcal{T}_\beta$ so that $i_{\beta\delta}(\nu) = \eta$. Then put $c^\delta_\eta = c^\beta_\nu$. This defines for all $\beta \leq \delta$ an s-indiscernible tree $(c^\beta_\eta)_{\eta \in \mathcal{T}_\eta}$ satisfying our requirements. □

{movingtms}

Lemma 4.12. *Suppose T is NSOP$_1$, $M \models T$, and $a \mathop{\smile\!\!\!\!\!\!\;|}^K_M b$. Then there is a tree Morley sequence $(a_i)_{i<\omega}$ which is Mb-indiscernible with $a_0 = a$.*

Proof. By Lemma 4.11, for arbitrarily large cardinals κ, there is a tree $(c_\eta)_{\eta \in \mathcal{T}_\kappa}$ which is spread out and s-indiscernible over M so that if $\eta \triangleright \nu$ and $\text{dom}(\nu) = \kappa$ then $c_\eta c_\nu \equiv_M ab$. Note that $\mathcal{T}' = \mathcal{T}_\kappa \setminus \{\nu \in \mathcal{T}_\kappa : \text{dom}(\nu) = \kappa\} = \{\eta \in \mathcal{T}_\kappa : \text{dom}(\eta) \subseteq [1, \kappa)\}$ is isomorphic to $\mathcal{T}_\kappa$. So we may enumerate $(c_\eta)_{\eta \in \mathcal{T}'}$ as $(d_\eta)_{\eta \in \mathcal{T}_\kappa}$. Note that for all $\eta \in \mathcal{T}_\kappa$, $d_\eta \equiv_M a$ and $d_{\zeta_\alpha} = c_{\zeta_{1+\alpha}}$ for all $\alpha < \kappa$. By Lemma 4.10, there is a Morley tree over M $(d'_\eta)_{\eta \in \mathcal{T}_\omega}$ so that for all $w \in [\omega]^{<\omega}$ there is $v \in [\kappa]^{<\omega}$ so that $(d_\eta)_{\eta \in \mathcal{T}_\kappa \restriction v} \equiv_M (d'_\eta)_{\eta \in \mathcal{T}_\omega \restriction w}$.

Let $p(x; a) = \text{tp}(b/Ma)$. We claim $\bigcup_{i<\omega} p(x; d'_{\zeta_i})$ is consistent. Given n, let $w = \{0, \ldots, n-1\}$. Find $v \in [\kappa]^{<\omega}$ so that $(d_\eta)_{\eta \in \mathcal{T}_\kappa \restriction v} \equiv_M (d'_\eta)_{\eta \in \mathcal{T}_\omega \restriction w}$. If $v = \{\alpha_0, \ldots, \alpha_{n-1}\}$, then for $i < n$ we have $d_{\zeta_\alpha} = c_{1+\zeta_\alpha}$. Then because $c_{\zeta_{1+\alpha_i}} c_{\zeta_0} \equiv_M ab$ for all $i < n$, we have $c_{\zeta_0} \models \bigcup_{i<n} p(x; d_{\zeta_{\alpha_i}})$. This shows $\bigcup_{i<n} p(x; d_{\zeta_{\alpha_i}})$ is consistent and hence $\bigcup_{i<n} p(x; d'_{\zeta_i})$ is consistent. The claim follows by compactness.

Let $b' \models \bigcup_{i<\omega} p(x; d'_{\zeta_i})$. Extract from $(d'_{\zeta_i})_{i<\omega}$ an Mb-indiscernible sequence $(a_i)_{i<\omega}$. As $(a_i)_{i<\omega} \equiv_M (d'_{\zeta_i})_{i<\omega}$, we know $(a_i)_{i<\omega}$ is a tree Morley sequence. By an automorphism, we may assume $b' = b$ and $a_0 = a$. □

{kimslemmafortms}

Proposition 4.13. *(Kim's lemma for tree Morley sequences) Suppose T is NSOP$_1$ and $M \models T$. The following are equivalent:*

(1) $\varphi(x; a)$ Kim-divides over M.
(2) For some tree Morley sequence $(a_i)_{i<\omega}$ over M with $a_0 = a$, $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.
(3) For every tree Morley sequence $(a_i)_{i<\omega}$ over M with $a_0 = a$, $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.

Proof. Suppose $(a_i)_{i<\omega}$ is a tree Morley sequence over M. Let $(a_\eta)_{\eta \in \mathcal{T}_\omega}$ be a Morley tree over M with $a_{\zeta_i} = a_i$. Let $\eta_i \in \mathcal{T}_\omega$ be the function with $\text{dom}(\eta_i) = [i, \omega)$ and

$$\eta_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Consider the sequence $I = (a_{\eta_{2i}}, a_{\zeta_{2i+1}})_{i<\omega}$. Because $(a_\eta)_{\eta \in \mathcal{T}_\omega}$ is a Morley tree over M, I is an M-indiscernible sequence. Moreover, by s-indiscernibility, $a_{\eta_0} \equiv_{MI_{>0}} a_{\zeta_0}$ and because it is a Morley tree, we have $a_{\zeta_0} \equiv_{MI_{>0}} a_{\zeta_1}$. Therefore $a_{\eta_0} \equiv_{MI_{>0}} a_{\zeta_1}$. By indiscernibility, for all i, we have $a_{\eta_{2i}} \equiv_{MI_{>i}} a_{\zeta_{2i+1}}$. Because $(a_\eta)_{\eta \in \mathcal{T}_\omega}$ is a spread out tree over M, $a_{\eta_i} \mathop{\smile\!\!\!\!\!\mid}^i_M a_{\eta_{<i}}$ for all i. Because $(a_{\eta_{2i}})_{i<\omega}$ is moreover an M-indiscernible sequence, it is a Morley sequence in some global M-invariant type. By NSOP$_1$, it follows that $\{\varphi(x; a_{\eta_{2i}}) : i < \omega\}$ is consistent if and only if $\{\varphi(x; a_{\zeta_{2i+1}}) : i < \omega\}$ is consistent: if exactly one of them is consistent, then we have SOP$_1$ by Proposition 2.4.

As $(a_{\eta_{2i}})_{i<\omega}$ is a Morley sequence in some global M-invariant type extending $\text{tp}(a/M)$, $\varphi(x; a)$ Kim-divides over M if and only if $\{\varphi(x; a_{\eta_{2i}}) : i < \omega\}$ is inconsistent. So we've shown $\varphi(x; a)$ Kim-divides if and only if $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent, by indiscernibility. This proves (1) $\implies$ (3), since if $(a_i)_{i<\omega}$ was a tree Morley sequence over M with $a_0 = a$ and $\{\varphi(x; a_i) : i < \omega\}$ consistent, then the above argument shows $\varphi(x; a)$ does not Kim-divide over M. This also proves (2) $\implies$ (1), since if $(a_i)_{i<\omega}$ is a tree Morley sequence over M with $a_0 = a$ and $\{\varphi(x; a_i) : i < \omega\}$ inconsistent then the above argument shows $\varphi(x; a)$ must Kim-divide over M. The direction (3) $\implies$ (2) is trivial. □

Proposition 4.14. *(Chain condition for tree Morley sequences) Suppose T is NSOP$_1$ and $M \models T$. If $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ and $I = (b_i)_{i<\omega}$ is a tree Morley sequence over M with $b_0 = b$, then there is $a' \equiv_{Mb} a$ so that $a' \mathop{\smile\!\!\!\!\!\mid}^K_M I$ and I is Ma'-indiscernible.* {tmschaincondition}

Proof. We follow the proof of Proposition 3.19 above. Given $n < \omega$, let $r(x; y_0, \ldots, y_{n-1}) = \text{tp}(a'; b_0, \ldots, b_{n-1}/M)$. By Lemma 4.9, $\langle (b_{kn+n-1}, b_{kn+n-2}, \ldots, b_{kn}) : k < \omega \rangle$ is a tree Morley sequence over M and, by indiscernibility,

$$a' \models \bigcup_{k<\omega} r(x; b_{kn+n-1}, b_{kn+n-2}, \ldots, b_{kn}).$$

By Proposition 4.13, this shows $a' \mathop{\smile\!\!\!\!\!\mid}^K_M b_{<n}$. □

Theorem 4.15. *(Symmetry) Suppose T is a complete theory. The following are equivalent:* {symmetrycharthm}

(1) T is NSOP$_1$

(2) $\mathop{\smile\!\!\!\!\!\mid}^K$ is symmetric over models: for any $M \models T$ and tuples a, b from $\mathbb{M}$, $a \mathop{\smile\!\!\!\!\!\mid}^K_M b \iff b \mathop{\smile\!\!\!\!\!\mid}^K_M a$

(3) $\mathop{\smile\!\!\!\!\!\mid}^K$ enjoys the following weak symmetry property: for any $M \models T$ and tuples a, b from $\mathbb{M}$, $a \mathop{\smile\!\!\!\!\!\mid}^i_M b$ implies $b \mathop{\smile\!\!\!\!\!\mid}^K_M a$.

Proof. (1) $\iff$ (3) is Proposition 3.20 and (2) $\implies$ (3) is immediate from the fact that $a \mathop{\smile\!\!\!\!\!\mid}^i_M b$ implies $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$.

(1) $\implies$ (2). Suppose T is NSOP$_1$. Assume towards contradiction that $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$ and $b \not\mathop{\smile\!\!\!\!\!\mid}^K_M a$. By Lemma 4.12, there is a tree Morley sequence over M with $a_0 = a$ which is Mb-indiscernible. Since $b \not\mathop{\smile\!\!\!\!\!\mid}^K_M a$, there is some $\varphi(x; a) \in \text{tp}(b/Ma)$ which Kim-divides over M. By Lemma 4.13, $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent. But $\models \varphi(b; a_i)$ for all $i < \omega$ by indiscernibility, a contradiction. □

5. The Independence Theorem

{itsection}

5.1. Strengthening the chain condition. Before beginning the proof of the independence theorem, we will observe that tree Morley sequences can be 'widened' by adding on an independent tuple to each tuple in the sequence. Similar operations are possible for Morley sequences in simple theories using transitivity and base monotonicity; in our context, we prove this by directly constructing an s-indiscernible and spread out tree.

{widening}

Proposition 5.1. *Assume T is NSOP$_1$ and $M \models T$. Suppose $a \mathop{\smile\!\!\!\!\!\!\,|}^K_M b$ and $(b_i)_{i<\kappa}$ is an Ma-indiscernible tree Morley sequence over M in $tp(b/M)$. Then there is a tree Morley sequence $\langle a_i b_i : i < \kappa \rangle$ so that*

$$a_i b_{\geq i} \equiv_M a_0 \overline{b}_{\geq 0},$$

for all $i < \kappa$.

Proof. By induction on $\alpha < \kappa$, we will build $(a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ and sequences $\langle b_{\alpha,\beta} : \alpha + 1 \leq \beta < \kappa \rangle$ so that

(1) For all $\eta \in \mathcal{T}_\alpha$, $a^\alpha_\eta \overline{b}^\alpha_{\trianglelefteq \eta} \overline{b}_\alpha \equiv_M a \langle b_\beta : \min \operatorname{dom}(\eta) \leq \beta < \kappa \rangle$.
(2) $\langle b_{\alpha,\beta} : \alpha + 1 \leq \beta < \kappa \rangle$ is $M(a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$-indiscernible.
(3) $(a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ is spread out over M and s-indiscernible over $M\overline{b}_\alpha$.
(4) If $\alpha < \beta$, then $(a^\beta_{i_{\alpha\beta}(\eta)}, b^\beta_{i_{\alpha\beta}(\eta)}) = (a^\alpha_\eta, b^\alpha_\eta)$ for all $\eta \in \mathcal{T}_\alpha$.

Note that $b^\alpha_{\trianglelefteq \eta}$ enumerates a sequence indexed by $[\min \operatorname{dom}(\eta), \alpha)$. Concatenation with the sequence $\overline{b}_\alpha$ yields a sequence indexed by $[\min \operatorname{dom}(\eta), \kappa)$. Our intention in (1) is that there is an automorphism in $\operatorname{Aut}(\mathbb{M}/M)$ sending this sequence to $\langle b_\beta : \beta \in [\alpha, \kappa) \rangle$ which moreover sends a^α_η to a.

For the base case, define $(a^0_\emptyset, b^0_\emptyset) = (a_0, b_0)$. This defines $(a^0_\eta, a^0_\eta)_{\eta \in \mathcal{T}_0}$. Define $\langle b_{0,\beta} : 1 \leq \beta < \kappa \rangle$ by $b_{0,\beta} = b_\beta$.

Suppose for some $\alpha < \kappa$, we are given $(a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ and $\overline{b}_\alpha$ satisfying the requirements. By (2), we have $\langle b_{\alpha,\beta} : \alpha + 1 \leq \beta < \kappa \rangle$ is $M(a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$-indiscernible so

$$(a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\!\,|}^K_M \overline{b}_\alpha,$$

since $\overline{b}_\alpha$ is a tree Morley sequence over M by (1). By symmetry, then,

$$\overline{b}_\alpha \mathop{\smile\!\!\!\!\!\!\,|}^K_M (a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}.$$

Therefore, there is an M-invariant Morley sequence $\langle (a^\alpha_{\eta,i}, b^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \omega \rangle$ which is $M\overline{b}_\alpha$-indiscernible with $(a^\alpha_{\eta,0}, b^\alpha_{\eta,0})_{\eta \in \mathcal{T}_\alpha} = (a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$. Choose some a_* so that $a_* \overline{b}_\alpha \equiv_M a \langle b_\beta : \alpha + 1 \leq \beta < \kappa \rangle$. By compactness, Ramsey, and an automorphism, we may assume that $\langle (a^\alpha_{\eta,i}, b^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \omega \rangle$ is $Ma_*\overline{b}_\alpha$-indiscernible. Define a tree $(c_\eta, d_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ by $(a^\alpha_\eta, b^\alpha_\eta) = (c_{i_{\alpha\alpha+1}(\eta)}, d_{i_{\alpha\alpha+1}(\eta)})$ and $(c_\emptyset, d_\emptyset) = (a_*, b_{\alpha,\alpha+1})$. Let $(a^{\alpha+1}_\eta, b^{\alpha+1}_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ be a tree s-indiscernible over $M\langle b_{\alpha,\beta} : \alpha + 2 \leq \beta < \kappa \rangle$ and locally based on $(c_\eta, d_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$. Finally, define $\langle b_{\alpha+1,\beta} : \alpha + 2 \leq \beta < \kappa \rangle$ to be an $M(a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$-indiscernible sequence locally based on $\langle b_{\alpha,\beta} : \alpha + 2 \leq \beta < \kappa \rangle$. This completes the construction at successor steps. By an automorphism, we may assume $(a^{\alpha+1}_{i_{\alpha\alpha+1}(\eta)}, b^{\alpha+1}_{i_{\alpha\alpha+1}(\eta)}) = (a^\alpha_\eta, a^\alpha_\eta)$ for all $\eta \in \mathcal{T}_\alpha$ so condition (4) is preserved.

If δ is a limit and we're given $(a^\alpha_\eta, b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ for all $\alpha < \delta$, define for all $\alpha < \delta$, $(a^\delta_{i_{\alpha\delta}(\eta)}, b^\delta_{i_{\alpha\delta}(\eta)})$ for all $\eta \in \mathcal{T}_\alpha$. Condition (4) guarantess that this is well-defined and, as $\mathcal{T}_\delta$ is the direct limit of the $\mathcal{T}_\alpha$, this defines $(a^\delta_\eta, b^\delta_\eta)$ for all $\eta \in \mathcal{T}_\delta$. By compactness and (4), we may easily find a sequence $\langle b_{\delta,\beta} : \delta \leq \beta < \kappa \rangle$ satisfying the requirements. □

Corollary 5.2. *(Strong Chain Condition) Suppose* $a \mathop{\smile\hskip-0.9em ^| \ }^K_M bc$ *and* $I = (c_i)_{i<\kappa}$ *is an* M*-invariant Morley sequence which is moreover* Mb*-indiscernible. Then there is* $a' \equiv_{Mbc} a$ *with* $a' \mathop{\smile\hskip-0.9em ^| \ }^K_M bI$ *and* I *is* Ma'*-indiscernible.*

Proof. By Proposition 5.1, we may extend I to a tree Morley sequence $J = (b_i, c_i)_{i<\kappa}$ with $b_0 = b$. By the chain condition for tree Morley sequences, there is $a' \equiv_{Mbc} a$ so that J is Ma'-indiscernible and $a' \mathop{\smile\hskip-0.9em ^| \ }^K_M J$. In particular, $a' \mathop{\smile\hskip-0.9em ^| \ }^K_M bI$. □

5.2. **The proof of the independence theorem.**

Fact 5.3. [CK12, Remark 2.16] Write $a \mathop{\smile\hskip-0.9em ^| \ }^u_A b$ to mean that $\text{tp}(a/Ab)$ is finitely satisfiable in A – the u is for "ultrafilter" as this is equivalent to asserting $\text{tp}(a/Ab) = \text{Av}(\mathcal{D}, Ab)$ for some ultrafilter $\mathcal{D}$ on A. The relation $\mathop{\smile\hskip-0.9em ^| \ }^u$ satisfies both left and right extension over models:

(1) If M is a model and $a \mathop{\smile\hskip-0.9em ^| \ }^u_M b$ then for all c, there is some $a' \equiv_{Mb} a$ so that $a' \mathop{\smile\hskip-0.9em ^| \ }^u_M bc$.
(2) If M is a model and $a \mathop{\smile\hskip-0.9em ^| \ }^u_M b$ then for all d, there is some $b' \equiv_{Ma} b$ so that $ad \mathop{\smile\hskip-0.9em ^| \ }^u_M b$.

The full independence theorem will be deduced from a weak independence theorem, which has an easy proof:

{reduction}

Proposition 5.4. *Assume T is NSOP$_1$. Then* $\mathop{\smile\hskip-0.9em ^| \ }^K$ *satisfies the following weak independence theorem over models: if* $M \models T$, $a \equiv_M a'$, $a \mathop{\smile\hskip-0.9em ^| \ }^K_M b$, $a' \mathop{\smile\hskip-0.9em ^| \ }^K_M c$ *and* $b \mathop{\smile\hskip-0.9em ^| \ }^u_M c$*, then there is* a'' *with* $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$ *and* $a'' \mathop{\smile\hskip-0.9em ^| \ }^K_M bc$.

Proof. Suppose T is NSOP$_1$ and fix $M \models T$ and tuples a, a', b, c so that $a \equiv_M a'$, $a \mathop{\smile\hskip-0.9em ^| \ }^K_M b$, $a' \mathop{\smile\hskip-0.9em ^| \ }^K_M c$ and $b \mathop{\smile\hskip-0.9em ^| \ }^u_M c$.

Claim: There is c' so that $ac' \equiv_M a'c$ and $a \mathop{\smile\hskip-0.9em ^| \ }^K_M bc'$.

Proof of claim: By symmetry, it suffices to find c' with $ac' \equiv_M a'c$ and $bc' \mathop{\smile\hskip-0.9em ^| \ }^K_M a$. Let $p(x; a') = \text{tp}(c/Ma')$. By invariance, we know $p(x; a)$ does not Kim-fork over M. We have to show

$$p(x; a) \cup \{\neg\varphi(x, b; a) : \varphi(x, y; a) \in L(Ma) \text{ Kim-divides over } M\}$$

is consistent. If not, then by compactness and Kim-forking = Kim-dividing, we must have

$$p(x; a) \vdash \varphi(x, b; a),$$

for some φ where $\varphi(x, y; a)$ Kim-divides over M. By symmetry, $b \mathop{\smile\hskip-0.9em ^| \ }^K_M a$, so there is some M-invariant Morley sequence $(a_i)_{i<\omega}$ with $a_0 = a$ which is moreover Mb-indiscernible. Then we have

$$\bigcup_{i<\omega} p(x; a_i) \vdash \{\varphi(x, b; a_i) : i < \omega\}.$$

As $p(x;a)$ does not Kim-fork over M, we know $\bigcup_{i<\omega} p(x;a_i)$ is consistent. But, by Kim's lemma for Kim-dividing, we know $\{\varphi(x,y;a_i) : i<\omega\}$ is inconsistent and *a fortiori* $\{\varphi(x,b;a_i) : i<\omega\}$ is inconsistent, a contradiction. So the given partial type is consistent. Let c' realize it. Then $ac' \equiv_M a'c$ and $c'b \mathop{\smile\!\!\!\!\!\!\mid}^K_M a$, which proves the claim. □

As $b \mathop{\smile\!\!\!\!\!\!\mid}^u_M c$, by left extension, there is $c'' \equiv_{Mb} c$ with $bc' \mathop{\smile\!\!\!\!\!\!\mid}^u_M c''$. Then by right extension and automorphism, we can choose some b'' so that $bc' \equiv_M b''c''$ and $bc' \mathop{\smile\!\!\!\!\!\!\mid}^u_M b''c''$. As $bc' \mathop{\smile\!\!\!\!\!\!\mid}^u_M b''c''$ and $bc' \equiv_M b''c''$, it follows that $(b''c'', bc')$ starts a Morley sequence I in some global M-finitely satisfiable (hence M-invariant) type. As $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M bc'$, we may, by the chain condition (Proposition 3.19) find some $a_* \equiv_{Mbc'} a$ so that I is Ma_*-indiscernible and $a_* \mathop{\smile\!\!\!\!\!\!\mid}^K_M I$. Then, we obtain $a_* \equiv_{Mb} a$, $a_*c'' \equiv_M a'c$, and $a_* \mathop{\smile\!\!\!\!\!\!\mid}^K_M bc''$. By construction, $c'' \equiv_{Mb} c$ so there is $\sigma \in \text{Aut}(\mathbb{M}/Mb)$ with $\sigma(c'') = c$. Then $\sigma(a_*) \mathop{\smile\!\!\!\!\!\!\mid}^K_M bc$, $\sigma(a_*) \equiv_{Mb} a$, and $\sigma(a_*) \equiv_{Mc} a'$, which shows that the weak independence theorem over models holds for T. □

{consistenttree}

Lemma 5.5. *Suppose T is NSOP$_1$, $M \models T$, and $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$. Fix an ordinal α and any $q \supseteq tp(b/M)$, a global M-invariant type. If $(b_\eta)_{\eta \in \mathcal{T}_\alpha}$ is a tree, spread out over M, so that, for all $\nu \in \mathcal{T}_\alpha$, $b_\nu \models q|_{Mb_{\rhd \nu}}$, then, writing $p(x;b)$ for $tp(a/Mb)$, we have*

$$\bigcup_{\eta \in \mathcal{T}_\alpha} p(x;b_\eta)$$

is consistent and non-Kim-forking.

Proof. Recall that ζ_β is the function in $\mathcal{T}_\alpha$ with domain $[\beta,\alpha)$ which is identically zero. Our assumption entails that $(b_{\zeta_\beta})_{\beta<\alpha}$ is a Morley sequence over M in q. By the chain condition, there is $a' \models \bigcup_{\beta<\alpha} p(x;b_{\zeta_\eta})$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_M (b_{\zeta_\beta})_{\beta<\alpha}$. By induction on $\beta<\alpha$, we will pick a_β so that

$$a_\beta \models \bigcup_{\eta \trianglerighteq \zeta_\beta} p(x;b_\eta) \cup \bigcup_{\beta<\gamma<\alpha} p(x;b_{\zeta_\gamma})$$

and $a_\beta \mathop{\smile\!\!\!\!\!\!\mid}^K_M (b_\eta)_{\eta \trianglerighteq \zeta_\beta} (b_{\zeta_\gamma})_{\beta<\gamma<\alpha}$. As $\mathcal{T}_\alpha = \bigcup_{\beta<\alpha}\{\eta \in \mathcal{T}_\alpha : \zeta_\beta \trianglelefteq \eta\}$, this suffices.

To start, put $a_0 = a'$. Now suppose a_β is given. Note $\zeta_{\beta+1} \frown \langle 0 \rangle = \zeta_\beta$, so we have

$$a_\beta \mathop{\smile\!\!\!\!\!\!\mid}^K_M (b_\eta)_{\eta \trianglerighteq \zeta_{\beta+1} \frown \langle 0 \rangle} (b_{\zeta_\gamma})_{\beta<\gamma<\alpha}.$$

As the tree is spread out over M, the sequence $\langle b_{\trianglerighteq \zeta_{\beta+1} \frown \langle i \rangle} : i<\omega \rangle$ is an M-invariant Morley sequence and, moreover, we know $(b_{\zeta_\gamma})_{\beta<\gamma<\alpha}$ is a Morley sequence over $M(b_{\trianglerighteq \zeta_{\beta+1} \frown \langle i \rangle})_{i<\omega}$ in q, which implies that $\langle b_{\trianglerighteq \zeta_{\beta+1} \frown \langle i \rangle} : i<\omega \rangle$ is $M(b_\gamma)_{\beta<\gamma<\alpha}$-indiscernible. By the strong chain condition, we may find $a_{\beta+1}$ so that

$$a_{\beta+1} \equiv_{M(b_\eta)_{\eta \trianglerighteq \zeta_{\beta+1} \frown \langle 0 \rangle} (b_{\zeta_\gamma})_{\beta<\gamma<\alpha}} a_\beta$$

and also $\langle b_{\trianglerighteq \zeta_{\beta+1} \frown \langle i \rangle} : i<\omega \rangle$ is $Ma_{\beta+1}$-indiscernible and

$$a_{\beta+1} \mathop{\smile\!\!\!\!\!\!\mid}^K_M ((b_\eta)_{\eta \trianglerighteq \zeta_{\beta+1} \frown \langle i \rangle})_{i<\omega} (b_{\zeta_\gamma})_{\beta<\gamma<\alpha}.$$

Unravelling, we have

$$a_{\beta+1} \models \bigcup_{\eta \trianglerighteq \zeta_{\beta+1}} p(x; b_\eta) \cup \bigcup_{\beta+1<\gamma<\alpha} p(x; b_{\zeta_\gamma})$$

and $a_{\beta+1} \downfree^K_M (b_\eta)_{\eta \trianglerighteq \zeta_{\beta+1}} (b_{\zeta_\gamma})_{\beta+1<\gamma<\alpha}$. This completes the successor step.

If $\delta < \alpha$ is a limit and we have a_β for all $\beta < \delta$, we may find an a_δ satisfying the requirements by compactness and finite character of Kim-dividing. This completes the proof. □

{zigzag}

Lemma 5.6. *Suppose the complete theory T is NSOP$_1$, $M \models T$ and $b \downfree^K_M b'$. Then for any global M-invariant type $q \supseteq tp(b/M)$, there is a TMS $(b_i, b'_i)_{i<\omega}$ starting with (b, b') so that*

(1) If $i \leq j$, then $b_i b'_j \equiv_M bb'$
(2) If $i > j$, then $b_i \models q|_{Mb_j}$.

Proof. Fix $q \supseteq \text{tp}(b/M)$ and let $p(x; b) = \text{tp}(b'/Mb)$. By recursion on α, we will construct trees $(c^\alpha_\eta, d^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ so that, for all α

(1) If $\eta \in \mathcal{T}_\alpha$, then
$$c^\alpha_\eta \models q|_{M c^\alpha_{\triangleright \eta} d^\alpha_{\triangleright \eta}}$$
(2) If $\eta \in \mathcal{T}_\alpha$, then
$$d^\alpha_\eta \models \bigcup_{\nu \trianglerighteq \eta} p(x; c^\alpha_\nu)$$
(3) $(c^\alpha_\eta, d^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ is spread out and s-indiscernible over M
(4) If $\beta < \alpha$ then $(c^\alpha_{i_{\beta\alpha}(\eta)}, d^\alpha_{i_{\beta\alpha}(\eta)}) = (c^\beta_\eta, d^\beta_\eta)$ for all $\eta \in \mathcal{T}_\beta$.

To start, define $(c^0_\emptyset, d^0_\emptyset) = (b, b')$. This defines $(c^0_\eta, d^0_\eta)_{\eta \in \mathcal{T}_0}$.

Now suppose given $(c^\alpha_\eta, d^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$. Let $\langle (c^\alpha_{\eta,i}, d^\alpha_{\eta,i}) : i < \omega \rangle$ be an M-invariant Morley sequence with $(c^\alpha_{\eta,0}, d^\alpha_{\eta,0})_{\eta \in \mathcal{T}_\alpha} = (c^\alpha_\eta, d^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$. Pick c_* so that

$$c_* \models q|_{M (c^\alpha_{\eta,i}, d^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha, i<\omega}}.$$

Then, by Lemma 5.5, we may choose d_* so that

$$d_* \models \bigcup_{\substack{\eta \in \mathcal{T}_\alpha \\ i<\omega}} p(x; c^\alpha_{\eta,i}) \cup p(x; c_*).$$

Define a tree $(e_\eta, f_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ by

$$\begin{aligned} (e_\emptyset, f_\emptyset) &= (c_*, d_*) \\ (e_{\langle i \rangle \frown \eta}, f_{\langle i \rangle \frown \eta}) &= (c^\alpha_{\eta,i}, d^\alpha_{\eta,i}). \end{aligned}$$

Finally, let $(c^{\alpha+1}_\eta, d^{\alpha+1}_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ be a tree s-indiscernible over M locally based on this tree. By an automorphism, we may assume that $c^{\alpha+1}_{i_{\alpha\alpha+1}(\eta)} = c^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$. This satisfies the requirements.

Finally, arriving to stage δ for δ limit, we simply define $(c^\delta_\eta, d^\delta_\eta)_{\eta \in \mathcal{T}_\delta}$ by stipulating $(c^\delta_{i_{\beta\delta}(\eta)}, d^\delta_{i_{\beta\delta}(\eta)}) = (c^\beta_\eta, d^\beta_\eta)$ for all $\beta < \delta$. By the coherence condition (4), this is well-defined, and satisfies the requirements. □

{itthmchar}

Theorem 5.7. *Suppose T is a complete theory. The following are equivalent:*

(1) T is NSOP$_1$

(2) $\mathop{\smile\!\!\!\!\!\!|}^K$ *satisfies the independence theorem over models: if* $M \models T$, $a \equiv_M a'$, $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!|}^K_M c$, *and* $b \mathop{\smile\!\!\!\!\!\!|}^K_M c$, *then there is* a'' *with* $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$ *and* $a'' \mathop{\smile\!\!\!\!\!\!|}^K_M bc$.

Proof. (2) $\implies$ (1) follows from [CR16, Theorem 5.1], using that $\mathop{\smile\!\!\!\!\!\!|}^i$ implies $\mathop{\smile\!\!\!\!\!\!|}^K$, together with symmetry for $\mathop{\smile\!\!\!\!\!\!|}^K$.

(1) $\implies$ (2): Assume T is NSOP$_1$. Suppose $M \models T$, $a \equiv_M a'$, and $a \mathop{\smile\!\!\!\!\!\!|}^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!|}^K_M c$ and $b \mathop{\smile\!\!\!\!\!\!|}^K_M c$. We must show there is a'' with $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$ and $a'' \mathop{\smile\!\!\!\!\!\!|}^K_M bc$. Let $p_0(x;b) = \text{tp}(a/Mb)$ and $p_1(x;c) = \text{tp}(a'/Mc)$. Suppose towards contradiction that $p_0(x;b) \cup p_1(x;c)$ Kim-forks over M. Let $q \supseteq \text{tp}(b/M)$ be a global type finitely satisfiable in M. In particular, q is M-invariant so, by Lemma 5.6, there is a tree Morley sequence over M, $(b_i, c_i)_{i\in\mathbb{Z}}$ so that

(1) If $i \leq j$, then $b_i c_j \equiv_M bc$
(2) If $i > j$, then $b_i \models q|_{Mc_j}$.

Then both $(b_{2i}, c_{2i+1})_{i\in\mathbb{Z}}$ and $(b_{2i}, c_{2i-1})_{i\in\mathbb{Z}}$ are tree Morley sequences over M. By (1), we know $p_0(x;b_0) \cup p_1(x;c_1)$ Kim-forks over M so

$$\bigcup_{i\in\mathbb{Z}} p_0(x;b_{2i}) \cup p_1(x;c_{2i+1})$$

is inconsistent. However, because $b_0 \mathop{\smile\!\!\!\!\!\!|}^u_M c_{-1}$ by (2), Proposition 5.4 gives that $p_0(x;b_0) \cup p_1(x;c_{-1})$ does not Kim-fork over M. Therefore

$$\bigcup_{i\in\mathbb{Z}} p_0(x;b_{2i}) \cup p_1(x;c_{2i-1})$$

is consistent. And this is a contradiction, as these two partial types are the same. This completes the proof. $\square$

Corollary 5.8. *Suppose* T *is NSOP*$_1$, $M \models T$, $b \equiv_M b'$ *and* $b \mathop{\smile\!\!\!\!\!\!|}^K_M b'$. *Then there is a tree Morley sequence over* M, *starting with* (b, b').

Proof. Let $p(x;b) = \text{tp}(b'/Mb)$. By induction on ordinals $\alpha \geq 1$, we will build trees $(b^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$ spread out and s-indiscernible over M so that

(1) $\nu \vartriangleleft \eta$ then $b^\alpha_\nu b^\alpha_\eta \equiv_M b'b$.
(2) $\bigcup_{\eta\in\mathcal{T}_\alpha} p(x;b^\alpha_\eta)$ does not Kim-fork over M.
(3) If $1 \leq \beta < \alpha$, then $b^\alpha_{i_{\beta\alpha}(\eta)} = b^\beta_\eta$.

To start, let $\bar{b} = (b_i)_{i<\omega}$ be an M-invariant Morley sequence - as $b \mathop{\smile\!\!\!\!\!\!|}^K_M b'$, we may assume this sequence is Mb'-indiscernible. By the chain condition, we have $b' \mathop{\smile\!\!\!\!\!\!|}^K_M \bar{b}$. Let $q(x;\bar{b}) = \text{tp}(b'/M\bar{b})$. By the Independence Theorem, there is $b'' \models p(x;b') \cup q(x;\bar{b})$ with $b'' \mathop{\smile\!\!\!\!\!\!|}^K_M b'\bar{b}$. Define $(b^1_\eta)_{\eta\in\mathcal{T}_1}$ by $b^1_\emptyset$ and $b^1_{\langle i\rangle} = b_i$. Then $(b^1_\eta)_{\eta\in\mathcal{T}_1}$ is spread out and s-indiscernible over M and clearly satisfies (1). Moreover, as

$$b'' \models \bigcup_{\eta\in\mathcal{T}_1} p(x;b^1_\eta)$$

and $b'' \mathop{\smile\!\!\!\!\!\!|}^K_M (b^1_\eta)_{\eta\in\mathcal{T}_1}$, (2) is satisfied as well.

Now suppose given $(b^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$. Let $\langle (b^\alpha_{\eta,i})_{\eta\in\mathcal{T}_\alpha} : i<\omega\rangle$ be an M-invariant Morley sequence with $(b^\alpha_{\eta,0})_{\eta\in\mathcal{T}_\alpha} = (b^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$. Choose $b'' \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M (b^\alpha)_{\eta\in\mathcal{T}_\alpha}$ with

$$b'' \models \bigcup_{\eta\in\mathcal{T}_\alpha} p(x;b^\alpha_\eta),$$

(this is possible by (2)). By the chain condition, we may assume the sequence $\langle (b^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha} : i<\omega\rangle$ is Mb''-indiscernible and that $b'' \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M (b^\alpha_{\eta,i})_{\eta\in\mathcal{T}_\alpha, i<\omega}$. Define a tree $(c_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ by $c_\emptyset = b''$ and $c_{\langle i\rangle\frown\eta} = b^\alpha_{\eta,i}$. Then let $(b^{\alpha+1}_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ be a tree which is s-indiscernible over M and locally based on $(c_\eta)_{\eta\in\mathcal{T}_\alpha}$. By an automorphism, we may assume that $b^{\alpha+1}_{i_{\alpha\alpha+1}(\eta)} = b^\alpha_\eta$ for all $\eta\in\mathcal{T}_\alpha$. This satisfies the requirements.

Finally, if δ is a limit and we are given $(b^\alpha_{\eta\in\mathcal{T}_\alpha})_{\eta\in\mathcal{T}_\alpha}$ for all $\alpha<\delta$, define $(b^\delta_{\eta\in\mathcal{T}_\delta}$ as follows: if $\eta\in\mathcal{T}_\delta$, choose any $\alpha<\delta$ and $\nu\in\mathcal{T}_\alpha$ so that $\eta = i_{\alpha\delta}(\nu)$. Then define $b^\delta_\eta = b^\alpha_\nu$. By the coherence condition, this is well-defined and clearly satisfies the requirements.

To conclude, let κ be big enough for Erdős-Rado and consider $(b^\kappa_\eta)_{\eta\in\mathcal{T}_\kappa}$ given by the above construction. Let $(b^\kappa_{\zeta_\alpha})_{\alpha<\kappa}$ be an enumeration of the all zero's path in the tree. Apply Erdős-Rado to find $(c_i)_{i<\omega}$ M-indiscernible and based on this sequence. By an automorphism, we may assume $c_0 = b$ and $c_1 = b'$. □

6. Forking and Witnesses

6.1. Basic properties of forking.

{forkingandidivdingdef}

Definition 6.1. (1) The formula $\varphi(x;b)$ *divides* over A if there is an A-indiscernible sequence $\langle b_i : i<\omega\rangle$ so that $\{\varphi(x;b_i) : i<\omega\}$ is inconsistent. A type $p(x)$ divides over A if it implies some formula that divides over A. Write $a \mathop{\smile\!\!\!\!\!\!\;\mid}^d_A B$ to mean that $\text{tp}(a/AB)$ does not divide over A.

(2) The formula $\varphi(x;b)$ *forks* over A if $\varphi(x;b)$ implies a finite disjunction $\bigvee_i \psi_i(x;c_i)$ where each $\psi_i(x;c_i)$ divides over A. A type $p(x)$ forks over A if it implies a formula which forks over A. We write $a \mathop{\smile\!\!\!\!\!\!\;\mid}^f_A B$ to mean that $\text{tp}(a/AB)$ does not fork over A.

(3) When M is a model, write $a \mathop{\smile\!\!\!\!\!\!\;\mid}^i_M b$ to mean $\text{tp}(a/Mb)$ extends to global M-invariant type.

The following facts about forking and dividing are easy and well-known – see, e.g., [GIL02] [Adl05].

Fact 6.2. The following are true with respect to an arbitrary theory:

(1) $a \mathop{\smile\!\!\!\!\!\!\;\mid}^d_A b$ if and only if, given any A-indiscernible sequence $I = \langle b_i : i<\omega\rangle$ with $b = b_0$, there is $a' \equiv_{Ab} a$ so that I is Aa'-indiscernible.

(2) $\mathop{\smile\!\!\!\!\!\!\;\mid}^f$ is an invariant ternary relation on small subsets satisfying:
  - (a) (Extension) If $a \mathop{\smile\!\!\!\!\!\!\;\mid}^f_A b$, then, for all c, there is $a' \equiv_{Ab} a$ so that $a' \mathop{\smile\!\!\!\!\!\!\;\mid}^f_A bc$.
  - (b) (Base Monotonicity) If $a \mathop{\smile\!\!\!\!\!\!\;\mid}^f_A bc$ then $a \mathop{\smile\!\!\!\!\!\!\;\mid}^f_{Ab} c$.
  - (c) (Left Transitivity) If $a \mathop{\smile\!\!\!\!\!\!\;\mid}^f_{Ab} c$ and $b \mathop{\smile\!\!\!\!\!\!\;\mid}^f_A c$ then $ab \mathop{\smile\!\!\!\!\!\!\;\mid}^f_A c$.

(3) For any model M,

$$a \mathop{\smile\!\!\!\!\!\!\;\mid}^i_M b \implies a \mathop{\smile\!\!\!\!\!\!\;\mid}^f_M b \implies a \mathop{\smile\!\!\!\!\!\!\;\mid}^K_M b.$$

As a warm-up to the theorem in the next subsection, we note that these properties easily give a weak form of transitivity for $\perp^K$:

Lemma 6.3. *Suppose* $a \perp^f_M bc$ *and* $b \perp^K_M c$. *Then* $ab \perp^K_M c$.

Proof. Assume $a \perp^f_M bc$ and $b \perp^K_M c$. As $b \perp^K_M c$ there is an M-invariant Morley sequence $I = (c_i)_{i<\omega}$ which is, moreover, Mb-indiscernible. By base monotonicity of $\perp^f$, $a \perp^f_{Mb} c$ so there is an Mab-indiscernible sequence $I' = (c'_i)_{i<\omega}$ with $I' \equiv_{Mbc} I$. Thus I' is an M-invariant Morley sequence with $c'_0 = c$ which is Mab-indiscernible. It follows that $ab \perp^K_M c$. □

6.2. **Morley Sequences.**

Definition 6.4. Suppose $M \models T$. An $\perp^K$-Morley sequence over M is an M-indiscernible sequence $\langle b_i : i < \omega \rangle$ satisfying $b_i \perp^K_M b_{<i}$. Likewise, an $\perp^f$-Morley sequence over M is an M-indiscernible sequence $\langle b_i : i < \omega \rangle$ satisfying $b_i \perp^f_M b_{<i}$.

{consistentindk} **Lemma 6.5.** *Suppose the complete theory* T *is NSOP*$_1$, $M \models T$, *and* $\varphi(x; b)$ *does not Kim-divide over* M. *Then for any* $\perp^K$*-Morley sequence* $\langle b_i : i < \omega \rangle$ *over* M *with* $b_0 = b$, $\{\varphi(x; b_i) : i < \omega\}$ *is non-Kim-forking over* M. *In particular, this set of formulas is consistent.*

Proof. By induction on n, we will show $\{\varphi(x; b_i) : i \leq n\}$ is non-Kim-forking over M. The case of $n = 0$ follows by hypothesis. Now suppose $\{\varphi(x; b_i) : i \leq n\}$ is non-Kim-forking over M. Fix $\sigma \in \text{Aut}(\mathbb{M}/M)$ with $\sigma(b_0) = b_{n+1}$. Let $a \models \{\varphi(x; b_i) : i \leq n\}$ with $a \perp^K_M b_{\leq n}$. Then $\sigma(a) \equiv_M a$ and $\models \varphi(\sigma(a); b_{n+1})$. We know $b_{n+1} \perp^K_M b_{\leq n}$ so by the Independence Theorem, there is a' with $a' \equiv_{Mb_{\leq n}} a$ and $a' \equiv_{Mb_{n+1}} \sigma(a)$ so that $a' \perp^K_M b_{\leq n+1}$. As $a' \models \{\varphi(x; b_i) : i \leq n+1\}$, this completes the induction. The lemma, then, follows by compactness. □

{kimslemmaforforking} **Theorem 6.6.** *Suppose the complete theory* T *is NSOP*$_1$ *and* $M \models T$. *The following are equivalent:*

1. *(1)* $\varphi(x; b)$ *Kim-divides over* M
2. *(2) For some* $\perp^f$*-Morley sequence* $(b_i)_{i<\omega}$ *over* M *with* $b_0 = b$, $\{\varphi(x; b_i) : i < \omega\}$ *is inconsistent.*
3. *(3) For every* $\perp^f$*-Morley sequence* $(b_i)_{i<\omega}$ *over* M *with* $b_0 = b$, $\{\varphi(x; b_i) : i < \omega\}$ *is inconsistent.*

Proof. (3) $\implies$ (1) $\iff$ (2) is immediate, as a Morley sequence in a global M-invariant type is, in particular, an $\perp^f$-Morley sequence and such sequences always exist.

Now we show (1) $\implies$ (3). Suppose not - assume that $\varphi(x; b)$ is a formula which Kim-divides over M, but there is some $\perp^f$-Morley sequence over M with $b_0 = b$ so that $\{\varphi(x; b_i) : i < \omega\}$ is consistent. By induction on n, we will construct a sequence $(b'_i)_{i\leq n}$ and an elementary chain $(N_i)_{i\leq n}$ so that

(1) For all $n < \omega$, $b_0 \dots b_n \equiv_M b'_0 \dots b'_n$
(2) For all $n < \omega$, $M \prec N_n \prec N_{n+1} \prec \mathbb{M}$
(3) For all $n < \omega$, $b'_n \perp^f_M N_n$
(4) For all $n < \omega$, $b'_n \in N_{n+1}$.

For the $n = 0$ case, set $b'_0 = b_0$ and $N_0 = M$. Now suppose we are given $(N_i)_{i \leq n}$ and $(b'_i)_{i \leq n}$. Let N_{n+1} be an arbitrary (small) elementary extension of N_n which contains b'_n. By invariance and extension of $\mathop{\smile\!\!\!\!\!\mid}^f$, we may choose some b'_{n+1} so that $b'_0 \dots b'_{n+1} \equiv_M b_0 \dots b_{n+1}$ and $b'_{n+1} \mathop{\smile\!\!\!\!\!\mid}^f_M N_{n+1}$. This completes the recursion.

Set $N = \bigcup_{i<\omega} N_i$.

Claim 1: For all $n < \omega$, $(b'_i)_{i \geq n} \mathop{\smile\!\!\!\!\!\mid}^f_M N_n$.

Proof of claim: Fix n. We will argue by induction on k that $b'_n \dots b'_{n+k} \mathop{\smile\!\!\!\!\!\mid}^f_M N_n$. For $k = 0$, this is by construction. Assume it has been proven for k. Note that $b'_{n+k+1} \mathop{\smile\!\!\!\!\!\mid}^f_M N_{n+k+1}$. Now N_n and $(b'_i)_{i \leq n+k}$ are contained in N_{n+k+1} so, in particular, we have $b'_{n+k+1} \mathop{\smile\!\!\!\!\!\mid}^f_M N_n b'_0 \dots b'_{n+k}$. By base monotonicity, we have

$$b'_{n+k+1} \mathop{\smile\!\!\!\!\!\mid}^f_{M b'_0 \dots b'_{n+k}} N_n.$$

This, together with the induction hypothesis, implies

$$b'_0 \dots b'_{n+k+1} \mathop{\smile\!\!\!\!\!\mid}^f_M N_n$$

by left-transitivity. The claim follows by finite character. □

Let $\mathcal{D}$ be any non-principal ultrafilter on $\{b'_i : i < \omega\}$ and $(c_i)_{i<\omega}$ be sequence chosen so that $c_i \models \text{Av}(\mathcal{D}, N c_{<i})$, i.e. a Morley sequence over N in the global $M(b'_i)_{i<\omega}$-invariant type $\text{Av}(\mathcal{D}, \mathbb{M})$.

Claim 2: $(c_i)_{i<\omega} \mathop{\smile\!\!\!\!\!\mid}^f_M N$.

Proof of claim: Suppose not. Then by finite character, there is l so that $(c_i)_{i<l} \not\mathop{\smile\!\!\!\!\!\mid}^f_M N$ so we choose some $\varphi(x_0, \dots, x_{l-1}; d) \in \text{tp}(c_0, \dots, c_{l-1}/N)$ which forks over M. Choose n so that $d \in N_n$. By definition of average type, we may find $i_0 > \dots > i_{l-1} > n$ so that $\mathbb{M} \models \varphi(b'_{i_0}, \dots, b'_{i_{l-1}}; d)$. Then $(b'_i)_{i \geq n} \not\mathop{\smile\!\!\!\!\!\mid}^f_M N_n$, contradicting Claim 1. □

Let $q \supseteq \text{tp}((c_i)_{i<\omega}/M)$ be a global M-invariant type. Let $\langle (c_{k,i})_{i<\omega} : k < \omega \rangle$ be a Morley sequence over M in q with $c_{0,i} = c_i$ for all $i < \omega$. By Claim 2, we know $(c_i)_{i<\omega} \mathop{\smile\!\!\!\!\!\mid}^f_M N$ so we may assume the sequence $\langle (c_{k,i})_{i<\omega} : k < \omega \rangle$ is N-indiscernible. We know that $\{\varphi(x; b_i) : i < \omega\}$ is consistent so $\{\varphi(x; b'_i) : i < \omega\}$ is consistent, and therefore $\{\varphi(x; c_{0,i}) : i < \omega\}$ is consistent. The sequence $(c_{0,i})_{i<\omega}$ is also an N-invariant Morley sequence so $\varphi(x; c_{0,0})$ does not Kim-divide over N. But as $c_{0,0} \equiv_M b$, $(c_{i,0})_{i<\omega}$ is an M-invariant Morley sequence over M, and $\varphi(x; b)$ Kim-divides over M, we know that $\{\varphi(x; c_{i,0}) : i < \omega\}$ is inconsistent.

Let $(\overline{d}_i)_{i<\omega}$ be an array, strongly indiscernible over N, locally based on $(\overline{c}_i)_{i<\omega}$. By Lemma 3.8, we have $(d_{i,0})_{i<\omega} \equiv_M (c_{i,0})_{i<\omega}$. Also, because $(\overline{c}_i)_{i<\omega}$ was taken to be N-indiscernible and $\overline{c}_0$ was an N-invariant Morley sequence, we know each $\overline{c}_i$ is an N-invariant Morley sequence, and therefore each $\overline{d}_i$ is an N-invariant Morley sequence. By choice of the array, $\{\varphi(x; d_{i,j}) : j < \omega\}$ is consistent for all i, so $\varphi(x; d_{i,0})$ does not Kim-divide over N. Also, we have $\{\varphi(x; d_{i,0}) : i < \omega\}$ is inconsistent. Thus, to derive a contradiction, it suffices by Lemma 6.5 to establish the following:

Claim 3: $(d_{i,0})_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence over N.

Proof of claim: As the $(d_{i,j})_{i,j<\omega}$ form a strongly indiscernible array over N, we know that for each $i < \omega$, $\overline{d}_i$ is an $N\overline{d}_{<i}$-indiscernible sequence. But it is also an N-invariant Morley sequence so $\overline{d}_{<i} \mathop{\smile\!\!\!\!\!\!\perp}^K_N d_{i,0}$. By symmetry, this yields in particular that $d_{i,0} \mathop{\smile\!\!\!\!\!\!\perp}^K_N d_{0,0} \dots d_{i-1,0}$. This proves the claim and completes the proof. □

6.3. **Witnesses.**

Definition 6.7. Suppose M is a model and $(a_i)_{i<\omega}$ is an M-indiscernible sequence.

(1) Say $(a_i)_{i<\omega}$ is a *witness* for Kim-dividing over M if, whenever $\varphi(x; a_0)$ Kim-divides over M, $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.
(2) Say $(a_i)_{i<\omega}$ is a *strong witness* to Kim-dividing over M if, for all n, the sequence $\langle (a_{n\cdot i}, a_{n\cdot i+1}, \dots, a_{n\cdot i+n-1}) : i < \omega\rangle$ is a witness to Kim-dividing over M.

Proposition 4.13 and Lemma 4.9 show that tree Morley sequences are strong witnesses for Kim-dividing. The following proposition shows the converse, giving a characterization of strong witnesses as exactly the tree Morley sequences.

{witnesschar}

Proposition 6.8. *Suppose T NSOP$_1$ and $M \models T$. Then $(a_i)_{i<\omega}$ is a strong witness for Kim-dividing over M if and only if $(a_i)_{i<\omega}$ is a tree Morley sequence over M.*

Proof. If $(a_i)_{i<\omega}$ is a tree Morley sequence, then $(a_{n\cdot i}, a_{n\cdot i+1}, \dots, a_{n\cdot i+(n-1)})_{i<\omega}$ is also a tree Morley sequence over M by Lemma 4.9. It follows that $(a_i)_{i<\omega}$ is a strong witness to Kim-dividing by Proposition 4.13.

For other other direction, suppose $(a_i)_{i<\omega}$ is a strong witness to Kim-dividing over M. Given an arbitrary cardinal κ, we may, by compactness, stretch the sequence to $(a_i)_{i<\kappa}$ which is still a strong witness to Kim-dividing over M. By recursion on $\alpha < \kappa$, we will construct trees $(a^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$ so that

(1) For all $i < \alpha$, $a^\alpha_{\zeta_i} = a_i$ and also $a^\alpha_{\zeta_\alpha} = a_\alpha$ for α successor.
(2) $(a^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$ is spread out over M and s-indiscernible over $M(a_i)_{i>\alpha}$.
(3) If $\alpha < \beta$, then $a^\alpha_\eta = a^\beta_{i_{\alpha\beta}(\eta)}$ for all $\eta \in \mathcal{T}_\alpha$.

For the case $\alpha = 0$, put $a^0_\emptyset = a_0$. This satisfies the demands. Suppose $(a^\beta_\eta)_{\eta\in\mathcal{T}_\beta}$ has been defined for all $\beta \leq \alpha$. By Ramsey, compactness, and an automorphism, we may assume $(a_i)_{i>\alpha}$ is $M(a^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$-indiscernible. As $I_{>\alpha} = (a_i)_{i>\alpha}$ is also a strong witness to Kim-dividing over M, we have

$$(a^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\!\perp}^K_M I_{>\alpha}.$$

Let $J = \langle (a^\alpha_{\eta,i})_{\eta\in\mathcal{T}_\alpha} : i < \omega\rangle$ be a Morley sequence in an M-invariant type with $a^\alpha_{\eta,0} = a^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$. By symmetry, $I_{>\alpha} \mathop{\smile\!\!\!\!\!\!\perp}^K_M (a^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$ so we may assume J is $MI_{>\alpha}$-indiscernible. Define the tree $(a^{\alpha+1}_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ by $a^{\alpha+1}_{\zeta_{\alpha+1}} = a_{\alpha+1}$ and $a^{\alpha+1}_{\langle i\rangle\frown\eta} = a^\alpha_{\eta,i}$ for all $\eta \in \mathcal{T}_\alpha$ and $i < \omega$. Note in particular, this definition gives $a^{\alpha+1}_{i_{\alpha\alpha+1}(\eta)} = a^{\alpha+1}_{0\frown\eta} = a^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$. The tree we just constructed is clearly spread out. By an automorphism, we may further assume $(a^{\alpha+1}_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ is s-indiscernible over $MI_{>\alpha+1}$. This completes the successor step.

Now suppose given $(a^\beta_\eta)_{\eta\in\mathcal{T}_\beta}$ for all $\beta < \delta$, where δ is a limit. Define $(a^\delta_\eta)_{\eta\in\mathcal{T}_\delta}$ by setting $a^\delta_{i_{\alpha\delta}(\eta)} = a^\alpha_\eta$ for all $\alpha < \delta$ and $\eta \in \mathcal{T}_\alpha$. Condition (3) guarantees that this is well-defined.

Taking κ to be sufficiently large, we may extract a Morley tree from the tree we just constructed - in particular, we may obtain a Morley tree $(b_\eta)_{\eta \in \mathcal{T}_\omega}$ so that $(b_{\zeta_i})_{i<\omega} \equiv_M (a_i)_{i<\omega}$. This shows that $(a_i)_{i<\omega}$ is a tree Morley sequence over M. □

{forkingimpliestree}

Corollary 6.9. *Suppose T is NSOP$_1$ and $M \models T$. An $\mathop{\smile\!\!\!\!\!\!\;|}^f$-Morley sequence over M is an tree Morley sequence.*

Proof. Suppose $(a_i)_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!\;|}^f$-Morley sequence over M.

Claim: For all $n < \omega$, $a_{>n} \mathop{\smile\!\!\!\!\!\!\;|}^f_M a_{\leq n}$.

We will argue as in Claim 1 in the proof of Theorem 6.6 above. By finite character, it suffices to show $a_{n+1} \dots a_{n+k+1} \mathop{\smile\!\!\!\!\!\!\;|}^f_M a_{<n}$ for all k. For $k = 0$, this is by definition of Morley sequence. Assuming for k, we have, by induction, $a_{n+1} \dots a_{n+k+1} \mathop{\smile\!\!\!\!\!\!\;|}^f_M a_{\leq n}$. We also have $a_{n+k+2} \mathop{\smile\!\!\!\!\!\!\;|}^f_M a_{\leq n+k+1}$ so by base monotonicity

$$a_{n+k+2} \mathop{\smile\!\!\!\!\!\!\;|}^f_{M a_{n+1} \dots a_{n+k+1}} a_{\leq n}.$$

By left-transitivity, we have

$$a_{n+1} \dots a_{n+k+1} a_{n+k+2} \mathop{\smile\!\!\!\!\!\!\;|}^f_M a_{\leq n},$$

which proves the claim.

By the claim, $\langle (a_{n \cdot i}, a_{n \cdot i+1}, \dots, a_{n \cdot i+n-1}) : i < \omega \rangle$ is an $\mathop{\smile\!\!\!\!\!\!\;|}^f$-Morley sequence over M, hence a witness to Kim-dividing over M by Theorem 6.6. This shows $(a_i)_{i<\omega}$ is a strong witness to Kim-dividing over M. By Proposition 6.8, $(a_i)_{i<\omega}$ is a tree Morley sequence over M. □

In any theory, if $(a_i)_{i<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!\;|}^f$-Morley sequence over A, then, as the proof of Corollary 6.9 shows, that $a_{>n} \mathop{\smile\!\!\!\!\!\!\;|}^f_A a_{\leq n}$ for all $n < \omega$. As base monotonicity and left-transitivity do not necessarily hold for $\mathop{\smile\!\!\!\!\!\!\;|}^K$, we give a Morley sequence with this stronger behavior a name:

Definition 6.10. Say the M-indiscernible sequence $(a_i)_{i<\omega}$ is a *total $\mathop{\smile\!\!\!\!\!\!\;|}^K$-Morley sequence* if $a_{>n} \mathop{\smile\!\!\!\!\!\!\;|}^K_M a_{\leq n}$ for all $n < \omega$.

Question 6.11. *Suppose T is NSOP$_1$, $M \models T$, and $I = (a_i)_{i<\omega}$ is a total $\mathop{\smile\!\!\!\!\!\!\;|}^K$-Morley sequence over M. Is I a tree Morley sequence over M?*

7. Characterizing NSOP$_1$ and Simple Theories

{mtsection}

7.1. **The Main Theorem.** Before continuing with the rest of the paper, we pause to take stock of what has been shown:

{mainthm}

Theorem 7.1. *The following are equivalent for the complete theory T:*

1. *T is NSOP$_1$*
2. *Ultrafilter independence of higher formulas: for every model $M \models T$, and ultrafilters $\mathcal{D}$ and $\mathcal{E}$ on M with $Av(\mathcal{D}, M) = Av(\mathcal{E}, M)$, $(\varphi, M, \mathcal{D})$ is higher if and only if $(\varphi, M, \mathcal{E})$ is higher*
3. *Kim's lemma for Kim-dividing: For every model $M \models T$ and $\varphi(x; b)$, if $\varphi(x; y)$ q-divides for some global M-invariant $q \supseteq tp(b/M)$, then $\varphi(x; y)$ q-divides for* every *global M-invariant $q \supseteq tp(b/M)$.*

(4) *Symmetry over models: for every $M \models T$, then $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M b$ if and only if $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M a$.*
(5) *Independence theorem over models: if $M \models T$, $a \equiv_M a'$, $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M b$, $a' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M c$, and $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M c$, then there is a'' with $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$ and $a'' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_M bc$.*

Proof. (1) $\iff$ (2) $\iff$ (3) is Theorem 3.14.
(1) $\iff$ (4) is Theorem 4.15.
(1) $\iff$ (5) is Theorem 5.7. □

7.2. **Simplicity within the class of NSOP$_1$ theories.**

{simplecosimpledef} **Definition 7.2.** [Che14, Section 6] Suppose $p(x)$ is a partial type over the set A.

(1) We say p is a *simple type* if there is no $\varphi(x;y)$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ and $k < \omega$ so that $\{\varphi(x; a_{\eta \frown \langle i \rangle}) : i < \omega\}$ is k-inconsistent for all $\eta \in \omega^{<\omega}$ and $\{\varphi(x; a_{\eta | i}) : i < \omega\}$ is consistent for all $\eta \in \omega^\omega$. Equivalently, $p(x)$ is simple if, whenever $B \supseteq A$, $q \in S(B)$, and $p \subseteq q$, then q does not divide over AB' for some $B' \subseteq B$, $|B'| \leq |T|$ (for the definition of dividing, see Definition 6.1 below).
(2) We say $p(x)$ is a *cosimple type* if there is no formula $\varphi(x;y) \in L(A)$ for which there exists $(a_\eta)_{\eta \in \omega^{<\omega}}$ and $k < \omega$ so that $\{\varphi(x; a_{\eta \frown \langle i \rangle}) : i < \omega\}$ is k-inconsistent for all $\eta \in \omega^{<\omega}$ and $\{\varphi(x; a_{\eta|i}) : i < \omega\}$ is consistent for all $\eta \in \omega^\omega$ and moreover $a_\eta \models p$ for all $\eta \in \omega^{<\omega}$.

{simpletype} **Proposition 7.3.** *Let $\pi(x)$ be a partial type over A.*

(1) *Assume that for any $\varphi(x;a)$ and $B \supseteq A$, $\pi \cup \{\varphi(x;a)\}$ divides over B if and only if $\pi \cup \{\varphi(x;a)\}$ Kim-divides over B. Then π is a simple type.*
(2) *Assume that if $B \supseteq A$, then for any a and for any $b \models \pi(x)$, $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^f_B b$ if and only if $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_B b$. Then π is a co-simple type.*

Proof. (1) Suppose π is not simple. Then by compactness, there is a formula $\varphi(x;y)$ over A and a tree $(a_\eta)_{\eta \in \omega^{<\omega+1}}$ s-indiscernible over A so that for some $k < \omega$

- For all $\eta \in \omega^{\omega+1}$, $\pi(x) \cup \{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega + 1\}$ is consistent
- For all $\eta \in \omega^{<\omega+1}$, $\{\varphi(x; a_{\eta \frown \alpha}) : \alpha < \omega\}$ is k-inconsistent.

Moreover we may assume $(a_{0^\alpha} : \alpha < \omega + 1)$ is an A-indiscernible sequence. Let $b \models \pi(x) \cup \{\varphi(x; a_{0^\alpha}) : \alpha < \omega + 1\}$. By Ramsey, compactness, and automorphism, we may assume $(a_{0^\alpha} : \alpha < \omega + 1)$ is Ab-indiscernible. Let $C = \{a_{0^\alpha} : \alpha < \omega\}$. Then s-indiscernibility implies $(a_{0^\omega \frown \beta} : \beta < \omega)$ is indiscernible over $A \cup C$ and $\{\varphi(x; a_{0^\omega \frown \beta}) : \beta < \omega\}$ is k-inconsistent by our assumption. As $b \models \varphi(x; a_{0^\omega \frown 0})$, we have $b \not\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^d_{AC} a_{0^\omega \frown 0}$. But by indiscernibility, $a_{0^\omega \frown 0} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_{AC} b$ so in particular $a_{0^\omega \frown 0} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_{AC} b$ and $b \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_{AC} a_{0^\omega \frown 0}$.

(2) We argue similarly. Suppose $(a_\eta)_{\eta \in \omega^{<\omega+1}}$ is a collection of realizations of π, forming a tree s-indiscernible over A, with respect to which $\varphi(x;y)$ witnesses the tree property. Let $a \models \{\varphi(x; b_{0^\alpha}) : \alpha < \omega + 1\}$. By Ramsey, compactness, and automorphism, we may assume $(b_{0^\alpha} : \alpha < \omega + 1)$ is a Ba-indiscernible sequence. Then we have $a \not\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^d_{A(b_{0^\alpha} : \alpha < \omega)} b_{0^\omega \frown 0}$ but $b_{0^\omega \frown 0} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^u_{A(b_{0^\alpha} : \alpha < \omega)} a$ so $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K_A b_{0^\omega \frown 0}$. □

{forkequalskimfork} **Corollary 7.4.** *The complete theory T is simple if and only if $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^f = \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^K$ over models.*

Definition 7.5. [YC14, Definition 2.5] We say $(a_i)_{i\in\kappa}$ is a *universal Morley sequence* in $p \in S(A)$ if

- $(a_i)_{i\in\kappa}$ is indiscernible with $a_i \models p$
- If $\varphi(x;y) \in L(A)$ and $\varphi(x;a_0)$ divides over A then $\{\varphi(x;a_i) : i \in \kappa\}$ is inconsistent.

{nomorley}

Proposition 7.6. *Suppose T is NSOP$_1$. Then T is simple if and only if, for any $M \models T$ and $p(x) \in S(M)$, there is a universal Morley sequence in p.*

Proof. If T is simple, then in any type $p(y) \in S(M)$, there is a $\perp^f$-Morley sequence in $p(y)$. By Kim's lemma for simple theories [Kim98, Proposition 2.1], this is a universal Morley sequence in p.

If T is not simple, then there is some formula $\varphi(x;b) \in L(Mb)$ which divides over M but does not Kim-divide over M, by Corollary forkequalskimfork. Suppose there is a universal Morley sequence in $\text{tp}(b/M)$ - by compactness we can take it to be $(b_i)_{i\in\mathbb{Q}}$ indexed by $\mathbb{Q}$. Then given $i \in \mathbb{Q}$, we have $b_{<i}$ is Mb_i-indiscernible so $b_{<i} \perp^d_M b_i$ so $b_i \perp^K_M b_{<i}$ by symmetry. So $(b_i)_{i\in\mathbb{Q}}$ is an $\perp^K$-Morley sequence. By Lemma 6.5, $\{\varphi(x;b_i) : i \in \mathbb{Q}\}$ is consistent. But $\varphi(x;b)$ divides over M and $(b_i)_{i\in\mathbb{Q}}$ is a universal Morley sequence so $\{\varphi(x;b_i) : i \in \mathbb{Q}\}$ is inconsistent. This is a contradiction. □

If $a \perp^K_M bb'$, it does not always make sense to ask if $a \perp^K_{Mb} b'$, since it is not always the case that $\text{tp}(b'/Mb)$ extends to a global Mb'-invariant type. This can occur, however, whenever Mb' is a model, for instance. Say $\perp^K$ satisfies base monotonicity *when appropriate* if, whenever $a \perp^K_M bb'$ and $\text{tp}(b'/Mb)$ extends to a global M-invariant type, then $a \perp^K_{Mb} b'$.

Proposition 7.7. *The NSOP$_1$ theory T is simple if and only if $\perp^K$ satisfies base monotonicity when appropriate.*

Proof. It T is simple, this is a well-known property of non-forking independence. On the other hand, suppose $\perp^K$ satisfies base monotonicity when appropriate. We will show that $\perp^K = \perp^d$ over models. It follows then that T is simple, by Corollary 7.4. So suppose towards contradiction that $a \perp^K_M b$ but $a \not\perp^d_M b$, witnessed by $\varphi(x;b) \in \text{tp}(a/Mb)$ and $I = (b_i)_{i<\omega+1}$ is an M-indiscernible sequence with $b_\omega = b$ and $\{\varphi(x;b_i) : i < \omega+1\}$ inconsistent. As $a \perp^K_M b$, we may, by extension, assume $a \perp^K_M I$. Now $\text{tp}(b/MI_{<\omega})$ is finitely satisfiable in $MI_{<\omega}$ by M-indiscernibility. So, by base monotonicity when appropriate, we have $a \perp^K_{MI_{<\omega}} b$. But stretching I to $(b_i)_{i<\omega+\omega}$, we have that $(b_{\omega+i})_{i<\omega}$ is a $MI_{<\omega}$-invariant Morley sequence (in the reverse order) in $\text{tp}(b/MI_{<\omega})$ and $\{\varphi(x;b_{\omega+i}) : i < \omega\}$ is inconsistent. So $a \not\perp^K_{MI_{<\omega}} b$, a contradiction. □

8. Examples

{examplesection}

8.1. **Kim-Pillay.** We are interested in explicitly describing $\perp^K$ in concrete examples. As in simple theories, this is most easily acheived by establishing the existence of an independence relation with certain properties and then deducing that, therefore, the relation coincides with $\perp^K$. The following theorem explains how this works.

{criterion}

Theorem 8.1. *Assume there is an $Aut(\mathbb{M})$-invariant ternary relation $\perp$ on small subsets of the monster $\mathbb{M} \models T$ such that it satisfies the following properties, for an arbitrary $M \models T$ and arbitrary tuples from $\mathbb{M}$.*

(1) *Strong finite character: if $a \not\perp_M b$, then there is a formula $\varphi(x, b, m) \in tp(a/bM)$ such that for any $a' \models \varphi(x, b, m)$, $a' \not\perp_M b$.*
(2) *Existence over models: $M \models T$ implies $a \perp_M M$ for any a.*
(3) *Monotonicity: $aa' \perp_M bb' \implies a \perp_M b$.*
(4) *Symmetry: $a \perp_M b \iff b \perp_M a$.*
(5) *The independence theorem: $a \perp_M b$, $a' \perp_M c$, $b \perp_M c$ and $a \equiv_M a'$ implies there is a'' with $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$ and $a'' \perp_M bc$*

Then T is NSOP$_1$ and $\perp$ strengthens *$\perp^K$ - i.e. if $M \models T$, $a \perp_M b$ then $a \perp^K_M b$. If, moreover, $\perp$ satisfies*

(6) *Witnessing: if $a \not\perp_M b$ witnessed by $\varphi(x; b)$ and $(b_i)_{i<\omega}$ is a global M-invariant type extending $tp(b/M)$, then $\{\varphi(x; b_i) : i < \omega\}$ is inconsistent.*

then $\perp = \perp^K$.

Proof. It was shown in [CR16] that if there is such a relation $\perp$, then T is NSOP$_1$. The proof there shows that if $\perp$ satisfies axioms (1)-(4), then $a \perp^u_M b$ implies $a \perp_M b$. Now suppose $a \perp_M b$. Let $p(x; b) = \text{tp}(a/Mb)$ and let q be a global coheir of $\text{tp}(b/M)$. By the independence theorem for $\perp$, $\bigcup_{i<\omega} p(x; b_i)$ is consistent. But then $a \perp^K_M b$. The "moreover" clause follows by definition of $\perp^K$. □

Remark 8.2. The condition (6) can be weakened to quantifying only over global *coheirs* of $\text{tp}(b/M)$ - this is sometimes slightly easier in practice.

Remark 8.3. Axioms (1)-(5) do not, by themselves, suffice to characterize $\perp^K$. See Remark 8.34 below.

8.2. **Combinatorial examples.** In this section, we study some combinatorial examples of NSOP$_1$ theories which are not simple. They are structures which encode a generic family of selector functions for an equivalence relation. The theories defined below provide a different presentation of the a theory defined by Džamonja and Shelah in [DS04] (where it was called T^*_{feq} – though this name is now typically reserved for a different theory) and later studied by Malliaris in [Mal12] (where it was called T^s). We give a family of theories T^*_n as n ranges over positive integers, but we will only be interested in the case of $n = 1, 2$. Among non-simple NSOP$_1$ theories, the theory T^*_1 is probably the easiest to understand, and we show that already T^*_1 witnesses many of the new phenomena in our context: with respect to this theory, we give explicit examples of formulas which divide but do not Kim-divide, formulas which fork and do not divide over models, and types which contain no universal Morley sequences.

We also use T^*_1 to answer a question of Chernikov from [Che14] concerning simple and co-simple types. A type is simple if no instance of the tree property is consistent with the type and a type is cosimple if the tree property cannot witnessed using parameters which realize the type (see Definition 7.2 above for the precise definition). For stability, no such distinction arises, but Chernikov was able to show that, in general, there are co-simple types which are not simple. In fact, examples can be found in the triangle-free random graph. It was asked in [Che14] if there can exist simple types which are not co-simple. We show the answer is yes already within the class of NSOP_1 theories.

Lastly, we use T_2^* to give a counter-example to transitivity for $\mathop{\smile\!\!\!\!\!\!\mid}^K$. Because Kim-dividing does not behave well with respect to changing the base, the normal formulation of transitivity doesn't make necessarily sense. Nonetheless, there is a natural way to formulate a version which does make sense. Suppose T is NSOP_1, $M \models T$ and both $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M bc$ and $b \mathop{\smile\!\!\!\!\!\!\mid}^K_M c$. Must it also be the case, then, that $ab \mathop{\smile\!\!\!\!\!\!\mid}^K_M c$? We show the answer is no.

For the remainder of this subsection, if A is a structure in some language and $X \subseteq A$, write $\langle X \rangle^A$ for the substructure of A generated by X. We write just $\langle X \rangle$ when A is the monster model.

For a natural number $n \geq 1$, let $L_n = \langle O, F, E, \text{eval} \rangle$ where O, F are sorts, E is a binary relation symbol, and eval is an $n+1$-ary function. The theory T_n will say

- O and F are sorts - O and F disjoint and the universe is their union.
- $E \subseteq O^2$ is an equivalence relation on O
- $\text{eval} : F^n \times O \to O$ is a function so that for all $f \in F^n$, $\text{eval}(f, -)$ is a function from O to O which is a selector function for E – more formally, for all $b \in O$, we have $E(\text{eval}(f, b), b)$ and if $b, b' \in O$ and $E(b, b')$ then we have
$$\text{eval}(f, b) = \text{eval}(f, b').$$

The letter F is for 'function' and O is for 'object' - we think of an element $f \in F$ as naming the function $\text{eval}(f, -)$. Let $\mathbb{K}_n$ be the class of finite models of T.

Lemma 8.4. *The class $\mathbb{K}_n$ is a Fraïssé class. Moreover, it is uniformly locally finite.*

Proof. HP is clear as the axioms of T_n are universal and, as we allow the empty structure, JEP will follow from SAP. So we show SAP. Suppose $A, B, C \in \mathbb{K}_n$ where $A \subseteq B, C$ and $B \cap C = A$. It suffices to define a L_n-structure with domain $D = B \cup C$, extending both B and C. Interpret O^D and F^D by $O^D = O^B \cup O^C$ and $F^D = F^B \cup F^C$. Let E^D be the equivalence relation generated by $E^B \cup E^C$. It follows that if $b \in B$, $c \in C$ and $(b, c) \in E^D$, then there is some $a \in A$ so that $(a, b) \in E^B$ and $(a, c) \in E^C$ and, moreover, (O^D, E^D) extends both (O^B, E^B) and (O^C, E^C) as equivalence relations.

We are left with interpreting eval^D. Let $\{a_i : i < k_0\}$ enumerate a collection of representatives for the E^A-classes in A. Then let $\{b_i : i < k_1\}$ and $\{c_i : i < k_2\}$ enumerate representatives for the E^B- and E^C-classes of elements not represented by an element of A, respectively. Then every element of O^D is equivalent to a unique element of

$$X = \{a_i : i < k_0\} \cup \{b_i : i < k_1\} \cup \{c_i : i < k_2\}.$$

Suppose $d \in X$. If $f \in (F^B)^n$, define $\text{eval}^D(f, d) = \text{eval}^B(f, d)$ if $d \in B$ and $\text{eval}^D(f, d) = d$ otherwise. Likewise, if $f \in (F^C)^n$ and $d \in C$, put $\text{eval}^D(f, d) = \text{eval}^C(f, c)$ if $c \in C$ and $\text{eval}^C(f, c) = c$ otherwise. If $f \in (F^D)^n \setminus ((F^B)^n \cup (F^C)^n)$, put $\text{eval}^D(f, d) = d$. This defines eval on $(F^D)^n \times X$. More generally, if $f \in (F^D)^n$ and $e \in O^D$, define $\text{eval}^D(f, e) = \text{eval}^D(f, d)$ for the unique $d \in X$ equivalent to e. This is well-defined as B and C agree on A and the D so-defined is clearly in $\mathbb{K}_n$.

Finally, note that a structure in $\mathbb{K}_n$ generated by k elements is obtained by applying $\leq k^n$ functions of the form $\mathrm{eval}(f,-)$ to $\leq k$ elements in O, so has cardinality $\leq k^{n+1}$. This shows $\mathbb{K}_n$ is uniformly locally finite. $\square$

It follows that there is a complete $\aleph_0$-categorical theory T_n^* extending T_n whose models have age $\mathbb{K}_n$. By the uniform local finiteness of $\mathbb{K}_n$, T_n^* has quantifier-elimination so T_n^* is the model completion of T_n. Let $\mathbb{M}_n \models T_n^*$ be a monster model.

Definition 8.5. Define a ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{*}$ on small subsets of $\mathbb{M}_n$ by: $a \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} b$ if and only if

(1) $\mathrm{dcl}(aC)/E \cap \mathrm{dcl}(bC)/E \subseteq \mathrm{dcl}(C)/E$
(2) $\mathrm{dcl}(aC) \cap \mathrm{dcl}(bC) \subseteq \mathrm{dcl}(C)$.

where $X/E = \{[x]_E : x \in X\}$ denotes the collection of E-classes represented by an element of X.

Remark 8.6. Note that by SAP, acl = dcl in T_n^*. From this, it is easy to check that $\mathop{\smile\!\!\!\!\!\!|}^{*}$ is just algebraic independence in $(T_n^*)^{eq}$.

encetheoremfor2functions}

Lemma 8.7. *The relation $\mathop{\smile\!\!\!\!\!\!|}^{*}$ satisfies the independence theorem over models: if $M \models T^*$, $a \equiv_M a'$, $a \mathop{\smile\!\!\!\!\!\!|}^{*}_{M} B$, $a' \mathop{\smile\!\!\!\!\!\!|}^{*}_{M} C$ and $B \mathop{\smile\!\!\!\!\!\!|}^{*}_{M} C$ then there is a'' with $a'' \equiv_{MB} a$, $a'' \equiv_{MC} a'$, and $a'' \mathop{\smile\!\!\!\!\!\!|}^{*}_{M} BC$.*

Proof. We may assume $M \subseteq B, C$ and that B and C are definably closed. Write $a = (d_0, \ldots, d_{k-1}, e_0, \ldots, e_{l-1})$ with $d_i \in F$ and $e_j \in O$ and likewise $a' = (d'_0, \ldots, d'_{k-1}, e'_0, \ldots, e'_{l-1})$. By uniform local finiteness, we may also assume that a is closed under the functions of L. Fix an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M}_n/M)$ with $\sigma(a) = a'$. Let $U = \{u_f : f \in \mathrm{dcl}(aB) \setminus B\}$ and $V = \{v_f : f \in \mathrm{dcl}(a'C) \setminus C\}$ denote collection of new formal elements with $u_h = v_{\sigma(h)}$ for all $h \in \langle aM \rangle \setminus B$. Let, then, a_* be defined by

$$a_* = (u_{d_0}, \ldots, u_{d_{k-1}}, u_{e_0}, \ldots, u_{e_{l-1}}) = (v_{d'_0}, \ldots, v_{d'_{k-1}}, v_{e'_0}, \ldots, v_{e'_{l-1}}).$$

We will construct by hand an L-structure D extending $\langle BC \rangle$ with domain $UV\langle BC \rangle$ in which $a^* \equiv_B a$, $a^* \equiv_C a'$ and $a^* \mathop{\smile\!\!\!\!\!\!|}^{*}_{M} BC$.

There is a bijection $\iota_0 : \mathrm{dcl}(aB) \to BU$ given by $\iota_0(b) = b$ for all $b \in B$ and $\iota_0(f) = u_f$ for all $f \in \mathrm{dcl}(aB) \setminus B$. Likewise, we have a bijection $\iota_1 : \mathrm{dcl}(a'C) \to CV$ given by $\iota_1(c) = c$ for all $c \in C$ and $\iota_1(f) = v_f$ for all $f \in \mathrm{dcl}(a'C) \setminus C$. The union of the images of these functions is the domain of the structure D to be constructed and their intersection is $\iota_0(\langle aM \rangle) = \iota_1(\langle a'M \rangle)$. Consider BU and CV as L_n-structures by pushing forward the structure on $\mathrm{dcl}(aB)$ and $\mathrm{dcl}(a'C)$ along ι_0 and ι_1, respectively. Note that $\iota_0|_{\langle aM \rangle} = (\iota_1 \circ \sigma)|_{\langle aM \rangle}$.

We are left to show that we can define an L_n-structure on $UV\langle BC \rangle$ extending that on BU, CV, and $\langle BC \rangle$ in such a way as to obtain a model of T_n^*. To begin, interpret the predicates by $O^D = O^{BU} \cup O^{CV} \cup O^{\langle BC \rangle}$ and $F^D = F^{BU} \cup F^{CV} \cup F^{\langle BC \rangle}$. Let E^D be defined to be the equivalence relation generated by E^{BU}, E^{CV}, and $E^{\langle BC \rangle}$. The interpretation of the predicates is well-defined since if f is an element of $\iota_0(\langle aM \rangle) = \iota_1(\langle a'M \rangle)$ then $\iota_0^{-1}(f)$ is in the predicate O if and only if $\iota_1^{-1}(f)$ is as well, and, moreover, it is easy to check that our assumptions on a, a', B, C entail that no pair of inequivalent elements in BU, CV, or $\langle BC \rangle$ become equivalent in D.

All that is left is to define the function eval^D extending $\mathrm{eval}^{BU} \cup \mathrm{eval}^{CV} \cup \mathrm{eval}^{\langle BC \rangle}$. We first claim that $\mathrm{eval}^{BU} \cup \mathrm{eval}^{CV} \cup \mathrm{eval}^{\langle BC \rangle}$ is a function. The intersection of the domains of the first two functions is (in a Cartesian power of) $\iota_0(\langle aM\rangle) = \iota_1(\langle aM\rangle)$. If b, b' are in this intersection, we must show

$$\mathrm{eval}^{BU}(b, b') = c \iff \mathrm{eval}^{CV}(b, b') = c.$$

Choose $b_0, b_0', c_0 \in \langle aM\rangle$ and $b_1, b_1', c_1 \in \langle a'M\rangle$ with $\iota_i(b_i, b_i', c_i) = (b, b', c)$. Then since $\iota_0 = \iota_1 \circ \sigma$ on $\langle aM\rangle$, we have

$$\mathbb{M}_n \models \mathrm{eval}(b_0, b_0') = c_0 \iff \mathbb{M}_n \models \mathrm{eval}(\sigma(b_0), \sigma(b_0')) = \sigma(c) \iff \mathbb{M}_n \models \mathrm{eval}(b_1, b_1') = c_1.$$

Since eval^{BU} and eval^{CV} are defined by pushing forward the structure on $\langle aB\rangle$ and $\langle a'C\rangle$ along ι_0 and ι_1, respectively, this shows that $\mathrm{eval}^{BU} \cup \mathrm{eval}^{CV}$ defines a function. Now the intersection of $\langle BC\rangle$ with $\iota_0(\langle aM\rangle^{\mathbb{M}})$ is BC and, by construction, all 3 functions agree on this set. So the union defines a function.

Choose a complete set of E^D-class representatives $\{d_i : i < \alpha\}$ so that if d_i represents an E^D-class that meets M then $d_i \in M$. If $e \in O^D$ is E^D-equivalent to some e' and (f, e') is in the domain of $\mathrm{eval}^{BU} \cup \mathrm{eval}^{CV} \cup \mathrm{eval}^{\langle BC\rangle^{\mathbb{M}}}$, define $\mathrm{eval}^D(f, e)$ to be the value that this function takes on on (f, e'). On the other hand, if $f \in (F^D)^n \setminus ((F^{BU})^n \cup (F^{CV})^n \cup (F^{\langle BC\rangle})^n)$ or e is not E^D-equivalent to any element on which $\mathrm{eval}^D(f, -)$ has already been defined, put $\mathrm{eval}^D(f, e) = d_i$ for the unique d_i E^D-equivalent to e. This now defines eval^D on all of $(F^D)^n \times O^D$ and, by construction, $\mathrm{eval}^D(f, -)$ is a selector function for E^D for all $f \in (F^D)^n$. This completes the construction of D and we've shown D is a model of T. By model-completeness and saturation, D embeds into $\mathbb{M}$ over BC. If we can show $a_* \perp^*_M BC$ in D, then this will be true for the image of a_* in D.

We have already argued that every E^D class represented by an element of a_* can only be equivalent to an element of B or C if it is equivalent to an element of M. Moreover, our construction has guaranteed $\langle a_* M\rangle^D \cap \langle BC\rangle^D \subseteq M$, where $\langle X\rangle^D$ denotes the substructure of D generated by X, so $a_* \perp^*_M BC$. □

{kimindependencefor2funct

Proposition 8.8. *The theory T_n^* is NSOP$_1$ and, moreover, if $M \models T_n^*$, then $a \perp^*_M b$ if and only if $a \perp^K_M b$.*

Proof. In Lemma 8.7, we showed $\perp^*$ satisfies the independence theorem over a model, and the other conditions (1)-(4) in Theorem 8.1 are clear for $\perp^*$. To show (6), notice that if $A \not\perp^*_M B$ with A, B definably closed and containing M, then either there is some $a \in A$ and $b \in B$ so that $\models E(a, b)$ and the E-class of b does not meet M or $a = b$ for some $b \notin M$. Then if $(b_i)_{i<\omega}$ is a Morley sequence in some global M-invariant $q \supseteq \mathrm{tp}(b/M)$, then both $\{E(x; b_i) : i < \omega\}$ or $\{x = b_i : i < \omega\}$ are 2-inconsistent. It follows that $\perp^* = \perp^K$ over models. □

{notcosimple}

Lemma 8.9. *Modulo T_1^*, the formula $O(x)$ axiomatizes a complete type over $\emptyset$ which is not co-simple.*

Proof. That $O(x)$ implies a complete type is clear from quantifier-elimination. In $O(\mathbb{M}_1)$, choose an array $(a_{\alpha,\beta})_{\alpha,\beta<\omega}$ of distinct elements so that, for all $\alpha < \alpha' < \omega$, given β, β', $\mathbb{M}_1 \models E(a_{\alpha,\beta}, a_{\alpha,\beta'})$ and $\mathbb{M} \models \neg E(a_{\alpha,\beta}, a_{\alpha',\beta'})$. Let $\varphi(x; y)$ be the formula $\mathrm{eval}(x, y) = y$. It is now easy to check

- For all functions $f : \omega \to \omega$, $\{\varphi(x; a_{\alpha,f(\alpha)}) : \alpha < \omega\}$ is consistent
- For all $\alpha < \omega$, $\{\varphi(x; a_{\alpha,\beta}) : \beta < \omega\}$ is 2-inconsistent,

so $\varphi(x;y)$ witnesses TP_2 with respect to parameters realizing $O(x)$. This shows $O(x)$ is not co-simple. □

Lemma 8.10. *Suppose $A \subseteq \mathbb{M}_1$. Then $acl(A) = dcl(A) = A \cup eval(F(A) \times O(A))$.*

Proof. The equality of $\mathrm{acl}(A)$ and $\mathrm{dcl}(A)$ follows from SAP for $\mathbb{K}_1$. The axioms of T_1^* imply that every term of L_1 is equivalent to one of the form x or $\mathrm{eval}(x,y)$, so $\mathrm{dcl}(A) = A \cup \mathrm{eval}(F(A) \times O(A))$. □

We will see that $\perp^*$ characterizes dividing when elements on the left-hand side come from O. The following lemma is the key ingredient in proving this:

{dividing}

Lemma 8.11. *Suppose $A \in \mathbb{K}_1$ and $A = \langle a, B\rangle^A$ for some $a \in O(A)$ and $B \in \mathbb{K}_1$, where $l(a) = 1$. Given a sequence $(B_i)_{i<N}$ of structures isomorphic to B over C where for $i \neq j$, $B_i \cap B_j = C$. Then provided (1) is true, then (2) holds as well:*

1. *a satisfies the following:*
   - *(a) $A \models a \neq b$ for all $b \in B \setminus C$.*
   - *(b) $A \models \neg E(a,b)$ for all $b \in B$ not E-equivalent in A to an element of C.*
2. *There is a structure $D \in \mathbb{K}_1$ and some $a' \in D$ so that*
   - *(a) $\langle (B_i)_{i<N}\rangle \subseteq D$.*
   - *(b) $\langle a', B_i\rangle^D \cong_C \langle a, B\rangle^A$ for all $i < N$.*

Proof. Suppose $A = \langle a, B\rangle^A$, $(B_i)_{i<N}$ and C are given as in statement, satisfying (1). If $a \in C$, the lemma follows from AP in $\mathbb{K}_1$ so assume it's not, and therefore $a \notin B$ by our assumption that $A \models a \neq b$ for all $b \in B \setminus C$. Moreover, we may assume $B_0 = B$. Note that the underlying set of A is $B \cup \{a\} \cup \mathrm{eval}(F(B), a)$. Let $X = \langle (B_i)_{i<N}\rangle$

Case 1: $A \models E(a,c)$ for some $c \in C$. In this case, the underlying set of A is $B \cup \{a\} \cup \mathrm{eval}(F(B), c) = B \cup \{a\}$. Let D be the extension of X with underlying set $X \cup \{a\}$ with relations interpreted so that $D \models a \in O \wedge E(a,c)$ and the function eval defined to extend eval^X and so that $\mathrm{eval}^D(d,a) = \mathrm{eval}^X(d,c)$ for all $d \in O^D$. It is easy to check that this satisfies (2).

Case 2: $A \models \neg E(a,c)$ for all $c \in C$. By our assumption that A satisfies (1), it follows that $A \models \neg E(a,b)$ for all $b \in B$ and hence the underlying set of A is the disjoint union of B and $\{a\} \cup \mathrm{eval}^A(F(B), a)$. Let $Y = \{a\} \cup \mathrm{eval}^A(F(B), a)$. We will define an L_1-structure extending X with underlying set $X \cup Y$. Interpret the sorts $F^D = F^X$ and $O^D = O^X \cup Y$. Define the equivalence relation so that $E^X \subset E^D$ and Y forms one E^D-class.

Fix for all $i < N$ a C-isomorphism $\sigma_i : B_i \to B_0$ (assume $\sigma_0 = \mathrm{id}_{B_0}$). Note that $F^X = \bigcup_{i<N} F^{B_i}$. Interpret eval^D to extend eval^X and so that, if $b \in F^{B_i}$ and $e \in Y$,

$$\mathrm{eval}^D(b,e) = \mathrm{eval}^A(\sigma_i(b), a).$$

This defines $D \in \mathbb{K}_1$ and, by construction, the map extending σ_i and sending $a \mapsto a$ induces an isomorphism $\langle a, B_i\rangle^D \to \langle a, B_0\rangle^D = A$ for all $i < N$. This completes the proof. □

{dividingmore}

Corollary 8.12. *If $a \in O(\mathbb{M}_1)$ and $l(a) = 1$, then $a \perp^d_E B$ if and only if $a \perp^*_E B$.*

Proof. If $a \not\perp^*_E B$ then clearly $a \not\perp^d_E B$, so we prove the other direction. Suppose $a \perp^*_E B$ and $a \not\perp^d_E B$ and we will get a contradiction. Suppose $\varphi(x;c,b)$ witnesses dividing, so $\varphi(x;c,b) \in \text{tp}(a/EB)$ with $c \in E$ and $b \in B$, and there is a E-indiscernible sequence $\langle b_i : i < \omega \rangle$ with $b_0 = b$ so that $\{\varphi(x;c,b_i) : i < \omega\}$ is k-inconsistent for some k. Let $C = \langle c \rangle^{\mathbb{M}}$, $B_i = \langle c, b_i \rangle^{\mathbb{M}}$ and $A = \langle a, b, c \rangle^{\mathbb{M}}$. As $a \perp^*_C B$, the structures A, C, and $(B_i)_{i<k+1}$ satisfy (1) of Lemma 8.11, and therefore there is $D \in \mathbb{K}_1$ and some $a' \in D$ so that $\langle (B_i)_{i<k+1} \subseteq D$ and $\langle a', B_i \rangle^D \cong \langle a, B \rangle^A$ for all $i < k+1$. By embedding D into $\mathbb{M}$ over $\langle (B_i)_{i<k+1} \rangle^D$ we see that, in $\mathbb{M}$, $\{\varphi(x;c,b_i) : i < k+1\}$ is consistent by quantifier-elimination. This is a contradiction. □

{artemQ}

Corollary 8.13. *The theory T_1^* is NSOP$_1$ and the formula $O(x)$ axiomatizes a complete type which is simple and not cosimple.*

Proof. Lemma 8.9 shows that $O(x)$ axiomatizes complete type which is not cosimple. To show $O(x)$ is simple, we have to show that $\perp^d$ satisfies local character on $O(x)$. So fix any $a \in \mathbb{M}_1$ with $\mathbb{M}_1 \models O(a)$ and any small set $B \subseteq \mathbb{M}_1$. We may suppose $B = \text{dcl}(B)$. Notice that $\text{dcl}(a) = a$. If $a \in B$ then $a \perp^*_a B$. If $a \notin B$ but $\mathbb{M} \models E(a,b)$ for some $b \in B$ then $a \perp^*_b B$. Finally, if a not E-equivalent to any element of B then $a \perp^*_\emptyset B$. Lemma **??** showed $a \perp^*_C B$ if and only if $a \perp^d_C B$ for any a with $\mathbb{M} \models O(a)$, so $\perp^d$ satisfies local character on O. Therefore O is simple. □

Remark 8.14. This answers Problem 6.10 of [Che14].

Remark 8.15. Given a model $M \models T_1^*$, one can consider the complete type $p(x)$ over M axiomatized by saying

- $O(x)$
- $\neg E(x,m)$ for all $m \in O(M)$
- $\text{eval}(x,m) \neq m$ for all $m \in O(M)$

In a similar fashion, one can check that this is simple, non-cosimple so, in particular, nothing is gained by working over a model. In fact, in this situation, we get a direct proof of the corollary, using Proposition 7.3, as we've shown that if $a \models p$, then $a \perp^d_M b$ if and only if $a \perp^K_M b$ so p is simple.

Proposition 7.6 above shows that in any non-simple NSOP$_1$ theory, there are types over models with no universal Morley sequences in them. The following gives an explicit example:

{nouniversal}

Proposition 8.16. *Given $M \models T$, there is a type $p \in S(M)$ with no universal Morley sequence.*

Proof. Pick $b \in O(\mathbb{M})$ not in M and let $p(x) = \text{tp}(b/M)$. Towards contradiction, suppose $(b_i)_{i<\omega}$ is a universal Morley sequence in p.

Case 1: $\mathbb{M} \models E(b_i, b_i)$ for all $i, j < \omega$.

The formula $E(x;b)$ divides over M: choose any M-indiscernible sequence $\langle c_i : i < \omega \rangle$ with $c_0 = b$ and $\neg E(c_i, c_{i+1})$ – then $\{E(x;c_i) : i < \omega\}$ is inconsistent. But $\{E(x;b_i) : i < \omega\}$ is consistent, a contradiction.

Case 2: $\mathbb{M} \models \neg E(b_i, b_j)$ for $i \neq j$. The formula $p_x(b) = b$ divides over M – chose any M-indiscernible sequence $\langle c_i : i < \omega \rangle$ with $E(c_i, c_j)$ for all i, j and $c_0 = b$. Then $\{p_x(c_i) = c_i : i < \omega\}$ is inconsistent (as for any a, the function p_a takes on only one value on elements of any equivalence class). But $\{p_x(b_i) = b_i : i < \omega\}$ is consistent, a contradiction. □

Proposition 8.17. *In T, forking does not equal dividing, even over models.*

Proof. Fix $M \models T$. Let $\varphi(x, y; z)$ be the formula $p_x(z) = z \vee E(y, z)$. Given any $b \in O(\mathbb{M})$ not in M, we claim the formula $\varphi(x, y; b)$ forks but does not divide over M. The proof of Proposition 8.16 shows that both $E(x, b)$ and $p_x(b) = b$ divide over M so $\varphi(x, y; b)$ forks over M. Given any M-indiscernible sequence $\langle b_i : i < \omega \rangle$ starting with b, either all b_i's lie in a single equivalence class, in which case $\{E(y, b_i) : i < \omega\}$ is consistent, or they all lie in different classes, in which case $\{p_x(b_i) = b_i : i < \omega\}$ is consistent. Either way, $\{\varphi(x, y; b_i) : i < \omega\}$ is consistent, so $\varphi(x, y; b)$ does not divide over M. □

The following problem was suggested by Artem Chernikov:

Question 8.18. *In an NSOP$_1$ theory, does forking = dividing over models for complete types? That is, if T is NSOP$_1$, $M \models T$, and $p(x) \in S(M)$ then must it be the case that $p(x)$ forks if and only if $p(x)$ divides?*

We note that graph-theoretic examples of theories for which forking and dividing are different, but coincide for complete types have been studied by Conant [Con14].

Finally, the following proposition gives a counter-example to the form of transitivity mentioned at the beginning of the subsection.

Proposition 8.19. *For any model $M \models T^*$, there are f, g, and c so that $f \mathop{\smile\!\!\!\!\!\!\mid}^K_M gc$, $g \mathop{\smile\!\!\!\!\!\!\mid}^K_M c$, and $fg \not\mathop{\smile\!\!\!\!\!\!\mid}^K_M c$.*

Proof. Given $M \models T$, choose any $c \in \mathbb{M} \backslash M$ in an E-class represented by an element m of M - let $\{m_i : i < \alpha\}$ enumerate a set of representatives for the remaining E-classes of M. Then choose distinct elements $f, g \in F$ so that

(1) $\text{eval}(f, g, m) = \text{eval}(g, f) = c$.
(2) $\text{eval}(f, h, m) = \text{eval}(h, f, m) = m$ and
$$\text{eval}(f, h, m_i) = \text{eval}(h, f, m_i) = m_i$$
for all $h \in F^M \cup \{f\}$.
(3) $\text{eval}(g, h, m) = \text{eval}(h, g, m) = m$ and
$$\text{eval}(g, h, m_i) = \text{eval}(h, g, m_i) = m_i$$
for all $h \in F^M \cup \{g\}$.

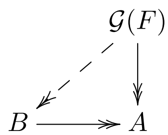
Then we have
$$\begin{aligned} \text{dcl}(fM) &= M \cup \{f\} \\ \text{dcl}(gM) &= M \cup \{g\} \\ \text{dcl}(cM) &= M \cup \{c\} \\ \text{dcl}(fgM) &= M \cup \{f, g, c\} \\ \text{dcl}(gcM) &= M \cup \{g, c\}. \end{aligned}$$

It follows that $\text{dcl}(fM) \cap \text{dcl}(gcM)$ and $\text{dcl}(gM) \cap \text{dcl}(cM)$ are contained in M so $f \mathop{\smile\!\!\!\!\!\!\mid}^*_M gc$ and $g \mathop{\smile\!\!\!\!\!\!\mid}^*_M c$. However, $c \in (\text{dcl}(fgM) \cap \text{dcl}(cM)) \setminus M$, showing $fg \not\mathop{\smile\!\!\!\!\!\!\mid}^*_M c$. □

8.3. **Frobenius Fields.**

Definition 8.20. Suppose F is a field.

(1) We say F is *pseudo-algebraically closed* (PAC) if every absolutely irreducible variety over F has an F-rational point.
(2) We say F is a *Frobenius field* if F is PAC and its absolute Galois group $\mathcal{G}(F)$ has the *embedding property*, that is, if $\alpha : \mathcal{G}(F) \to A$ and $\beta : B \to A$ are continuous epimorphisms and B is a finite quotient of $\mathcal{G}(F)$, then there is a continuous epimorphism $\gamma : \mathcal{G}(F) \to B$ so that $\beta \circ \gamma = \alpha$ as in the following diagram:



The free profinite group on countably many generators $\hat{F}_\omega$ has the embedding property so the ω-free PAC fields are Frobenius fields. However, there are many others - see, e.g., [FJ08, 24.6].

Definition 8.21. Suppose G is a profinite group. Let $\mathcal{N}(G)$ be the collection of open normal subgroups of G. We define

$$\mathcal{S}(G) = \coprod_{N \in \mathcal{N}(G)} G/N.$$

Let L_G the language with a sort X_n for each $n \in \mathbb{Z}^+$, two binary relation symbols $\leq, C$, and a ternary relation P. We regard $\mathcal{S}(G)$ as an L_G-structure in the following way:

- The coset gN is in sort X_n if and only if $[G : N] \leq n$.
- $gN \leq hM$ if and only if $N \subseteq M$
- $C(gN, hM) \iff N \subseteq M$ and $gM = hM$.
- $P(g_1N_1, g_2N_2, g_3N_3) \iff N_1 = N_2 = N_3$ and $g_1g_2N_1 = g_3N_1$.

Note that we do not require that the sorts be disjoint.

Fact 8.22. Given a field F, we write $\mathcal{G}(F)$ for its absolute Galois group. Suppose K and L are fields which are both regular extensions of the field E. Then if $K \equiv_E L$ then $\mathcal{S}(\mathcal{G}(K)) \equiv_{\mathcal{S}(\mathcal{G}(E))} \mathcal{S}(\mathcal{G}(L))$.

Proposition 8.23. *Suppose T is an NSOP$_1$ complete theory extending the theory of fields and $F \models T$. Assume $a \mathop{\smile\hspace{-0.9em}\mid}^K_F b$. Then the fields $A = acl(Fa)$ and $B = acl(Fb)$ satisfy the following conditions:* {kimimpliesweak1}

(1) A and B are linearly disjoint over F
(2) F^* is a separable extension of AB
(3) $acl(AB) \cap A^sB^s = AB$.

Proof. In [Cha99, Theorem 3.5], Chatizdakis proves (1)-(3) for an *arbitrary* theory of fields under the assumption that $a \mathop{\smile\hspace{-0.9em}\mid}^f_F b$. She deduces from $a \mathop{\smile\hspace{-0.9em}\mid}^f_F b$ that there is an F-indiscernible heir sequence $(B_i)_{i<\omega}$, i.e. an F-indiscernible sequence with $B_{<i} \mathop{\smile\hspace{-0.9em}\mid}^u_F B_i$ for all i, so that $AB_i \equiv_F AB$ for all i. She then proves that (1)-(3) follow from the existence of such a sequence. Note, however, that if T is NSOP$_1$, this follows merely from the assumption $a \mathop{\smile\hspace{-0.9em}\mid}^K_M b$ since an heir sequence $(B_i)_{i<\omega}$

is necessarily an $\perp^K$-Morley sequence (enumerated in reverse) so, if $p(x;B) = \mathrm{tp}(A/B)$, then $\bigcup_{i<\omega} p(x;B_i)$ is consistent, by Lemma [**?**]. Let A' be a realization. Moving the sequence by an automorphism, we may assume that $A' = A$ and $B_0 = B$. □

Remark 8.24. Note (1) and (2) are equivalent to saying $A \perp^{SCF}_F B$.

{kimimpliesweak2} **Lemma 8.25.** *Suppose F is a Frobenius field. If $A = acl(A)$, $B = acl(B)$ contain F and $A \perp^K_M B$ then $\mathcal{S}(\mathcal{G}(A)) \perp^f_{\mathcal{S}(\mathcal{G}(F))} \mathcal{S}(\mathcal{G}(B))$ in $Th(\mathcal{S}(\mathcal{G}(F)))$.*

Proof. Chatzidakis [C$^+$98] shows that the Galois group $\mathcal{S}(\mathcal{G}(F))$ is ω-stable. Let $(B_i)_{i<\omega}$ be a Morley sequence in a global type finitely satisfiable in F extending $\mathrm{tp}(B/F)$. As $A \perp^K_M B$, we may assume $(B_i)_{i<\omega}$ is A-indiscernible. Then $(\mathcal{S}(\mathcal{G}(B_i)))_{i<\omega}$ is a Morley sequence in a global type finitely satisfiable in $\mathcal{S}(\mathcal{G}(F))$ which is moreover $\mathcal{S}(\mathcal{G}(A))$-indiscernible. This implies $\mathcal{S}(\mathcal{G}(A)) \perp^K_{\mathcal{S}(\mathcal{G}(F))} \mathcal{S}(\mathcal{G}(B))$. As $\mathrm{Th}(\mathcal{S}(\mathcal{G}(F))$ is simple, this implies $\mathcal{S}(\mathcal{G}(A)) \perp^f_{\mathcal{S}(\mathcal{G}(F))} \mathcal{S}(\mathcal{G}(B))$ by CITE. □

Fix a field F and let SCF denote the complete theory of which F^s is a model.

Definition 8.26. Suppose $A = \mathrm{acl}(A)$, $B = \mathrm{acl}(B)$, and $C = \mathrm{acl}(C)$ in the field F. We say A is *weakly independent* from B over C if

(1) $A \perp^{\mathrm{SCF}}_C B$
(2) $\mathcal{S}(\mathcal{G}(A)) \perp^f_{\mathcal{S}(\mathcal{G}(F))} \mathcal{S}(\mathcal{G}(B))$, where $\perp^f$ denotes non-forking independence in $\mathrm{Th}(\mathcal{S}(\mathcal{G}(\mathcal{F})))$

Extend this to arbitrary tuples by stipulating a is *weakly independent* from b over c if and only if $\mathrm{acl}(a,c)$ is weakly independent from $\mathrm{acl}(b,c)$ over $\mathrm{acl}(c)$.

{zoeIT} **Theorem 8.27.** [Cha02, Theorem 6.1] *Let F be a Frobenius field, sufficiently saturated, and $E = acl(E)$ a subfield of F. Assume, moreover, that $acl(\mathcal{S}(\mathcal{G}(E))) = \mathcal{S}(\mathcal{G}(E))$ and if the degree of imperfection of F is finite, that E contains a p-basis of F. Assume that the tuples a, b, c_1, c_2 of F satisfy:*

(1) a and c_1 are weakly independent over E, b and c_2 are weakly independent over E, $c_1 \equiv_E c_2$
(2) $acl(Ea)$ and $acl(Eb)$ are SCF-independent over E.

Then there is c realizing $tp(acl(Ea)) \cup tp(c_2/acl(Eb))$ such that c and $acl(Eab)$ are weakly independent over E.

Theorem 8.28. *Suppose F is a Frobenius field and a, b are tuples from an elementary extension of F. Then $a \perp^K_F b$ if and only if a and b are weakly independent over F.*

Proof. First, we note that weak independence satisfies axioms (1)-(5) in Theorem 8.1. Non-forking independence always satisfies strong finite character, existence over models, and monotonicity so (1)-(3) are satisfied for $\perp^f$ in the theory of separably closed fields and $\mathrm{Th}(\mathcal{S}(\mathcal{G}(F)))$. As both of these theories are stable, $\perp^f$ is also symmetric. This shows (1)-(4). Condition (5) follows from Theorem [**?**]. By theorem 8.1, this shows that $\mathrm{Th}(F)$ is NSOP_1 and if a is weakly independent from b over F, then $a \perp^K_F b$. Conversely, if $a \perp^K_F b$, then $\mathrm{acl}(aF) \perp^{SCF}_F \mathrm{acl}(bF)$ by Proposition 8.23 and $\mathcal{S}(\mathcal{G}(\mathrm{acl}(aF))) \perp^f_{\mathcal{S}(\mathcal{G}(F))} \mathcal{S}(\mathcal{G}(\mathrm{acl}(bF)))$ by Lemma 8.25. This shows a and b are weakly independent over F. □

8.4. **Vector spaces.** The theory T_∞ will denote the theory of infinite dimensional vector spaces over an algebraically closed field equipped with a generic bilinear form. The language is two-sorted: there is a sort V for the vector space, with the language of abelian groups on it, a sort K for the field, equipped with the ring language, a function $K \times V \to V$ for the action of scalar multiplication, and a function $[,] : V \times V \to K$ for the bilinear form. We will use T_∞ to refer to both the theory where the form is symmetric and where it is alternating, as this choice makes no difference for our analysis below. In this subsection, we will write $\mathbb{M} \models T_\infty$ for a fixed monster model of T_∞.

Fact 8.29. Given a set $X \subseteq \mathbb{M}$, write X_K for the field points of X and X_V for the vector space points of X. For Y a set of vectors, write $\langle Y \rangle$ for the $\mathbb{M}_K$-span of V.

(1) T_∞ eliminates quantifiers after expanding the vector space sort with an n-ary predicate θ_n interpreted so that $\models \theta_n(v_0, \ldots, v_{n-1})$ if and only if $v_0, \ldots, v_{n-1}$ are linearly independent for all $n \geq 2$.
(2) For any set $A \subseteq \mathbb{M}$, the field points of $\text{dcl}(A)$ are the field generated by A_K, $\{[a,b] : a, b \in A_V\}$, and for each n, and every set $\{\alpha_0, \ldots, \alpha_{n-1}\}$ such that there are $v_0, \ldots, v_n \in A_V$ with $\mathbb{M} \models \theta_n(v_0, \ldots, v_{n-1})$ and $v_n = \alpha_0 v_0 + \ldots + \alpha_{n-1} v_{n-1}$. Then the vector space points of $\text{dcl}(A)$ are the $(\text{dcl}(A))_K$-span of A_V. The field points of $\text{acl}(A)$ are the algebraic closure of $(\text{dcl}(A))_K$ and the vector space points of $\text{acl}(A)$ are the $(\text{acl}(A))_K$-span of A_V.

Definition 8.30. Suppose $A \subseteq B$ and c is a singleton. Let $c \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{A} B$ be the assertion that $(\text{dcl}(cA))_K \mathop{\smile\!\!\!\!\!\!|}^{ACF}_{(\text{dcl}(A))_K} (\text{dcl}(B))_K$ and one of the following holds:

(1) $c \in \mathbb{M}_K$
(2) $c \in \langle A \rangle$
(3) $c \notin \langle B \rangle$ and $[c, B]$ is Φ-independent over A,

where '$[c, B]$ is Φ-independent over A' means that whenever $\{b_0, \ldots, b_{n-1}\}$ is a linearly independent set in $B_V \cap (\mathbb{M}_V \setminus \langle A \rangle)$ then the set $\{[c, b_0], \ldots, [c, b_{n-1}]\}$ is algebraically independent over the compositum of $(\text{dcl}(B))_K$ and $(\text{dcl}(Ac))_K$.

By induction, for $c = (c_0, \ldots, c_m)$ define $c \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{A} B$ by

$$c \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{A} B \iff (c_0, \ldots, c_{m-1}) \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{A} B \text{ and } c_m \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{Ac_0 \ldots c_{m-1}} Bc_0 \ldots c_{m-1}.$$

Fact 8.31. [Gra99, Theorem 12.2.2] [CR16] The relation $\mathop{\smile\!\!\!\!\!\!|}^{\Gamma}$ is automorphism invariant, symmetric, and transitive. Moreover, it satisfies extension, strong finite character, and stationarity over a model. Consequently, T_∞ is NSOP_1.

{kimchar}

Proposition 8.32. *Suppose $M \models T_\infty$. Then if $A = acl(A)$, $B = acl(B)$ and $A \cap B \supseteq M$, then $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} B$ if and only if $A \cap B = M$.*

Proof. Suppose M is a model, $A = \text{acl}(A)$, $B = \text{acl}(B)$, and $A \cap B \subseteq M$. Let $C = \text{acl}(AB)$ and let $(C_i)_{i<\omega}$ be an M-invariant Morley sequence over M with $C_0 = C$. Fix $\sigma \in \text{Aut}(\mathbb{M}/M)$ with $\sigma(C_i) = C_{i+1}$ for all $i < \omega$. By restricting the sequence $(C_i)_{i<\omega}$ to a subtuple, we obtain an M-invariant Morley sequence $(B_i)_{i<\omega}$ with $B_0 = B$. Let $D = \text{acl}((B_i)_{i<\omega})$. Let $\tilde{K} = (\text{acl}((C_i)_{i<\omega}))_K$. Let $\{u_i : i < \alpha\}$ be a basis for M_V. Let $\{v_i : i < \beta\}$ complete this set to a basis for A_V and let $(w_{0,j})_{j<\gamma}$ complete it to a basis for $(B_0)_V$, then let $(w_{i,j})_{j<\gamma}$ be the set of vectors completing $\{u_i : i < \alpha\}$ to a basis for $(B_i)_V$ corresponding to the $(w_{0,j})_{j<\beta}$ - i.e. $w_{i,j} = \sigma^i(w_{0,j})$. By our assumptions, $\{u_i : i < \alpha\} \cup \{v_i : i < \beta\} \cup \{w_{i,j} : i < \omega, j < \gamma\}$ is a set of linearly independent vectors in $\mathbb{M}_V$. Let $\tilde{V}$ be the $\tilde{K}$-vector space with basis $\{u_i : i < \alpha\} \cup \{v_i : i < \beta\} \cup \{w_{i,j} : i < \omega, j < \gamma\}$. To define the model $N = (\tilde{V}, \tilde{K})$, we are left with defining the form on $\tilde{V}$ - for this it suffices to define the form on a basis. First, interpret the form so that N extends the structure on D - i.e.

$$[u_i, u_{i'}]^N = k \iff [u_i, u_{i'}]^D = k$$
$$[u_i, w_{i',j}]^N = k \iff [u_i, w_{i',j}]^D = k$$
$$[w_{i,j}, w_{i',j'}]^N = k \iff [w_{i,j}, w_{i',j'}]^D = k.$$

And likewise, interpret the structure so that it extends the structure on A - i.e.

$$[u_i, v_{i'}]^N = k \iff [u_i, v_{i'}]^A = k.$$

Then finally, we interpret the form so that the structure generated by AB_i does not depend on i: put $[v_i, w_{0,j}]^N = k \iff [v_i, w_{0,j}]^C = k$ and set

$$[v_i, w_{i',j}]^N = \begin{cases} k & \text{if } [v_i, w_{0,j}]^C = k \in A \\ \sigma^{i'}(k) & \text{if } [v_i, w_{0,j}]^C = k \notin A \end{cases}$$

This defines N. By quantifier-elimination, there is an embedding $\iota : N \to \mathbb{M}$ over D into $\mathbb{M}$. Let $A' = \iota(A)$. By quantifier-elimination, we have $AB_0 \equiv_M A'B_i$ for all i. This shows $\text{tp}(A/B)$ does not Kim-divide over M. □

Proposition 8.33. *Suppose $M \models T_\infty$. Then*

(1) $a \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{M} b \implies a \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} b$.

(2) If a and b are singletons and $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} b \iff a \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{M} b$.

(3) There are a and b so that $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} b$ and $a \not\mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{M} b$.

Proof. (1) Suppose $a \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{M} b$. By transitivity of $\mathop{\smile\!\!\!\!\!\!|}^{\text{ACF}}$, $(\text{dcl}(aM))_K \mathop{\smile\!\!\!\!\!\!|}^{\text{ACF}}_{M_K} (\text{dcl}(bM))_K$ so

$$\text{acl}(aM)_K \mathop{\smile\!\!\!\!\!\!|}^{\text{ACF}}_{M} (\text{acl}(bM))_K$$

since the field points of the algebraic closure of any set X are just the field-theoretic algebraic closure of $(\text{dcl}(X))_K$. Similarly, transitivity of independence for vector spaces forces $\langle (aM)_V \rangle \cap \langle (bM)_V \rangle \subseteq \langle M \rangle$. This shows $\text{acl}(aM) \cap \text{acl}(bM) \subseteq M$ so $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} b$. by Proposition 8.32.

(2) By (1), it suffices to show $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} b \implies a \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{M} b$. As $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} b$, we know $\text{acl}(aM)_K \cap \text{acl}(bM)_K \subseteq M_K$ so, in particular,

$$\text{acl}(aM)_K \mathop{\smile\!\!\!\!\!\!|}^{ACF}_{M_K} \text{acl}(bM)_K.$$

Likewise, $\text{acl}(aM)_V \cap \text{acl}(bM)_V \subseteq M_V$ so $\langle (aM)_V \rangle \cap \langle (bM)_V \rangle \subseteq \langle M_V \rangle$. If a and b are singletons, then condition (3) in the definition of $\mathop{\smile\!\!\!\!\!\!|}^{\Gamma}$ is trivially satisfied, so $a \mathop{\smile\!\!\!\!\!\!|}^{\Gamma}_{M} b$.

(3) Given any $M \models T_\infty$, choose two vectors $b_1, b_2 \in \mathbb{M}_V$ that are $\mathbb{M}_K$-linearly independent over M. By model-completeness, we can find some vector a so that $\text{acl}(aM) \cap \text{acl}(b_1 b_2 M) \subseteq M$, so $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} b_1 b_2$, and also $[a, b_1] = [a, b_2]$. Then we clearly

have $\{[a, b_1], [a, b_2]\}$ algebraically dependent, as they are equal, hence $a \not\perp\!\!\!\perp^{\Gamma}_{M} b_1 b_2$. □

{notequal}

Remark 8.34. This observation implies that axioms (1)-(5) in Theorem 8.1 do not suffice to characterize $\perp\!\!\!\perp^{K}$, since $\perp\!\!\!\perp^{\Gamma}$ satisfies these axioms and $\perp\!\!\!\perp^{\Gamma} \neq \perp\!\!\!\perp^{K}$.

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