Frege's influence on modern practice of mathematics and logic

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Overview

1. Checking proofs

2. Hales' proof of the Kepler conjecture

3. Computerization of mathematical proofs

4. Frege's influence

Introduction

How do we come to see that a mathematical argument is correct?

- Prove it, then
- check whether the proof provided uses only given assumptions, already known facts, admitted axioms and inference rules.

Introduction

- However, many officially published work contains (*un*)detected errors.
- Still this process is considered generally reliable.

Introduction

There are however cases where this seemingly obvious process has difficulties to work.

Hales' proof of the Kepler conjecture

- The Kepler conjecture
  - No arrangement of equally sized spheres filling space has a greater average density than that of the cubic close packing (face-centered cubic) and hexagonal close packing arrangements. The density of these arrangements is around $\pi/3\sqrt{2} \simeq 0.7404$.

Hales' proof of the Kepler conjecture

- Hales' proof in August 1998
  - the proof consisted of 300 pages of texts and
  - 3 gigabytes of computer programs, data and results.

- Submitted to AMS
  - after 5 years of refereeing process
  - the panel of 12 referees were 99% certain of the correctness of the proof.
  - AMS published the text proofs only.

Hales' proof of the Kepler conjecture

What does "99% certainty" mean in mathematics?

Hales' proof of the Kepler conjecture

What was the problem?

Hales' proof of the Kepler conjecture

- H. Geuvers made an interesting comments on the refereeing process.
- Hales needed to prove that 1039 complicated inequalities hold.
- He used computer programs that verified the inequalities.
- The referees had problems with his approach:
  - verifying the inequalities themselves by hand would be impossible
  - one week per inequality is still 25 man years of work.
- They did not considered to verify the computer programs Hales used.

Worse cases

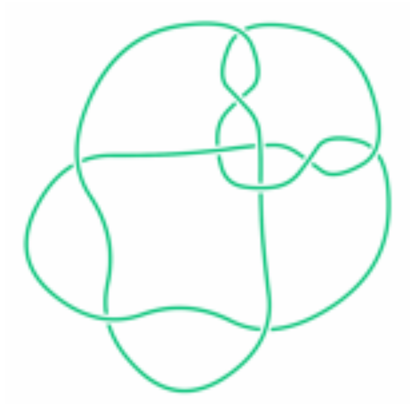
There are even cases in which some wrong statements were considered to be proved for a long period of time.

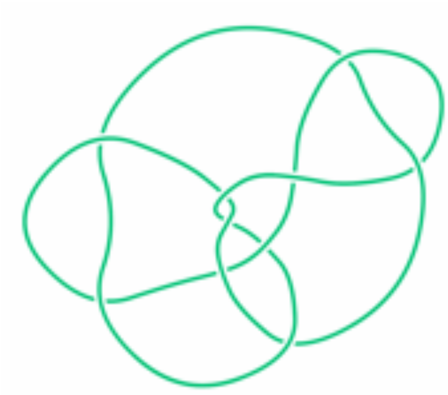
Worse cases 1

- In 1821, Cauchy published a proof that
  - a convergent sum of continuous functions is always continuous.
- In 1826, Abel found purported counterexamples in the context of Fourier series, arguing that Cauchy's proof had to be incorrect.
- In the modern language, what Cauchy proved is that a uniformly convergent sequence of continuous functions has a continuous limit.
- The failure illustrates the importance of distinguishing between different types of concepts.

Worse cases 2

- In the mathematical theory of knots, the Perko pair, named after Kenneth Perko, found in 1973, is a pair of entries in classical knot tables that actually represent the same knot.
- The Perko pair gives a counterexample to a theorem claimed by Little in 1900 that the writhe of a reduced diagram of a knot is an invariant.





Worse cases 3

- In 1933's paper "*On the decision problem for the functional calculus of logic*", Gödel claimed that the decidability of for a certain class of formulas can be shown.
- This claim was believed to be true for more than thirty years.
- But Aanderaa showed in the mid-1960s that Gödel's proof would not actually work if the formulas contained equality.
- Finally, in 1984 Goldfarb proved that the class mentioned by Gödel was not decidable.

Response

Mathematicians seem to have recognized the unreliability of checking process.

Response example

- In 2000 the Clay Mathematics Institute (CMI) announced million dollar prizes for the solution of seven *Millennium Problems*.
- But there are conditions according to which the prize would be awarded
  - two years after the appearance of the solution in a *refereed mathematics publication of worldwide repute;*
  - and after *general acceptance in the mathematics community*.
- But why wait two years? What does the ``*general acceptance in the mathematics community*" mean?
- Still these two conditions prove against the reliability of the traditional checking process.

Suggested solutions

- People like Doron Zeilberger suggest ways to improve the process.
- *In his blog post "If You Want Mathematical Truth, You Better Pay For It!"*

  Zeilberger suggests two ways:

  - Computerization!
  - Abandon the habit of anonymous refereeing and pay for it.

Computerization of mathematical proofs

- Example: Again Hales' proof of the Kepler conjecture
- In 2004, Hales himself announced his intention to have formal version of his original proof.
- His aim was to remove any remaining uncertainty about the validity of his proof by creating a formal proof that can be verified by automated proof checking software, that is by some computer programs.
- His intention was then realized through a project called Flyspeck on 10th August 2014, 10 years after his announcement.
- He used the *HOL Light* and *Isabelle* proof assistants.

Computerization of mathematical proofs

What does it mean to have a *formal version of proofs*?

Computerization of mathematical proofs

- In 2009 paper, *Proof assistants: History, ideas and future*, Geuvers gives a detailed and kind explanation of the basic ideas of proof assistants, targeting mathematicians without any background in computer science.
- I am not sure of that usual mathematicians, even logicians, would understand the details of the paper.
- But when one investigates some interest, then it would not be so difficult.

How mathematicians work

- In order to understand how proof assistants like HOL Light and Isabelle function as mathematicians work, it is necessary to understand
  - how mathematicians set up a theory and
  - how they define and prove mathematical properties.

Foundations of mathematics

- Around the turn of the 20th century mathematics and logicians started to intensively investigate the foundation of mathematics.
- The main motivation was to provide mathematics with
  - rigorous languages and
  - axiomatic systems

  where ordinary mathematical arguments can be represented and proved.

Frege's approach

- Gottlob Frege's main concern was twofold:
  - Firstly, whether arithmetical judgments can be proved in a purely logical manner.
  - Secondly, how far one could go in arithmetic by merely using the laws of logic.
- In *Begriffsschrift* (1879), he first invented a special kind of language where statements can be proved as true based only upon general logical laws and definitions.
- In the two volumes of *Basic Laws of Arithmetic* (1893, 1903) he used his system to provide a formal system where a system for second order arithmetic can be built up.
- Although this system is known to be inconsistent, it contains all the essential steps necessary to prove the fundamental propositions of arithmetic based on an axiom system.

Frege's influence 1

- What Frege does is to provide a *formal method for correct inferences of truth conditions* in his symbolic language without any supplementary intuitive reasoning.
- His work was a trigger for considering mathematical systems as axiomatic ones:
  - Peano's *The principles of arithmetic* (1889),
  - Hilbert's *Grundlagen der Geometrie* (1903),
  - Whitehead and Russell's *Principia Mathematica* (1910-1913),
  - Zermelo's axiomatic set theory of 1908,
  - Gentzen's Natural Deduction (1934) and Sequence Calculus (1934/35)
  - Church's type theory of 1940,
  - Martin-Löf type theory around 1980,
  - etc.
- Cf. van Heijennoort's *From Frege to Gödel* (1967).

Foundation for Proof assistants

- Mizar (1973~)
  - Tarski–Grothendieck set theory with classical logic
- PVS (1992~)
  - A classical, typed higher-order logic
- HOL family (HOL4, HOL Light, ProofPower)
  - A classical higher-order logic
- Isabelle
  - Zermelo-Fraenkel set theory (ZFC), higher-order logic
- Coq
  - Calculus of Inductive Constructions (CIC)
- Agda
  - Unified Theory of Dependent Types (UTT)

Frege's influence 2

- An important issue in the formalization of mathematical proofs
  - how to deal with variable binding
- Frege already suggested a solution in *Begriffsschrift* (1879).
  - distinguishing between free and bound variables

$$(a+b)\,c = a\,c + b\,c \quad \text{vs} \quad \forall\mathfrak{a}\,\forall\mathfrak{b}\,\forall\mathfrak{c}\,[\,(\mathfrak{a}+\mathfrak{b})\mathfrak{c} = \mathfrak{a}\ \mathfrak{c} + \mathfrak{b}\ \mathfrak{c}\,].$$

- Gentzen (1934) and Prawitz (1965) followed the same solution.
- This idea has got a name, namely Coquand-McKinna-Pollack style *locally named representation*.
- This representation style is the most natural technique in the domain of *formal proofs* although it is considered not so efficient.
- Our recent work has revealed a new aspect of this approach and showed that it still could be used in an efficient way under some conditions.